Chapter 1. Static Games of Complete Information.

Lec 1. Introduction

- Methodology of Game Theory → deal with strategic settings with many individuals

  - Key assumption: rationality. →
    - In economics, the standard form of rationality means that a decision maker chooses an action that yields maximum (expected) utility among all possible actions, given the decision maker's information.
    - In game theory, a player is **rational** if the player chooses an action that maximizes his expected payoff, given the player's beliefs about opponents' strategy choices.

- We focus on how players should behave in a certain sense rather than how they do behave.
- In this course, we focus on non-cooperative game theory framework.
  ↳ treat all players' action as individual actions.
  ↳ determined by a player herself.
- Different types of games
  - static vs dynamic — Static: one-shot, simultaneous-move
  - complete information vs incomplete information — Complete information: each player's payoff function is common knowledge among all players.

- 共有知识 Mutual knowledge : an event E is known by all players.
- 共同知识 Common knowledge : all players know E, all players know that they all know E …
  (我知道你知道我知道 ……) (要求能循环无限次)

- Four types of games
  1. Static games of complete information
  2. Dynamic games of complete information
  3. Static games of incomplete information
  4. Dynamic games of incomplete information
- Four corresponding solution concepts
  1. Nash equilibrium
  2. Subgame-perfect Nash equilibrium
  3. Bayesian Nash equilibrium
  4. Perfect Bayesian equilibrium

Lec 2. Normal-form Games

e.g. Prisoners' Dilemma → Static games of complete information
→ each player's payoff function is common knowledge among all players.
↳ one shot, simultaneous move.

- Normal-form representation:
  1. the players in the game;
  2. the strategies available to each player;
  3. the payoff received by each player for each combination of strategies that could be chosen by the players.

Def. The **normal-form** (also called **strategic-form**) representation of an n-player game specifies the players' **strategy spaces** $S_1, \dots, S_n$ and their **payoff functions** $u_1, \dots, u_n$. We denote this game by

$$G = \{S_1, \dots, S_n; u_1, \dots, u_n\}.$$

Let $(s_1, \dots, s_n)$ be a combination of strategies, one for each player. Then $u_i(s_1, \dots, s_n)$ is the payoff to player i if for each $j = 1, \dots, n$, player j chooses strategy s_j.

→ the payoff of a player depends not only on his own action, but also on the actions of others.
→ Interdependence (strategic interaction).

- In general, when there are only two players and each player has a finite number of strategies, then the payoff functions can be represented in a bi-matrix.
- The bi-matrix need not be symmetric, e.g.,

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\text{Player 2}} \\ & & L & R \\ \hline & U & u_1(U,L), u_2(U,L) & u_1(U,R), u_2(U,R) \\ \text{Player 1} & M & u_1(M,L), u_2(M,L) & u_1(M,R), u_2(M,R) \\ & D & u_1(D,L), u_2(D,L) & u_1(D,R), u_2(D,R) \\ \hline \end{array}$$

Prisoners' Dilemma

- For Example 1, the normal-form representation is $G = \{S_1, S_2; u_1, u_2\}$:
  - $S_1 = S_2 = \{D, C\}$, where D means "Defect" (不招供), and C means "Confess" (招供)
  - $u_1(D, D) = -1, u_1(D, C) = -9, u_1(C, D) = 0, u_1(C, C) = -6$
  - $u_2(D, D) = -1, u_2(D, C) = 0, u_2(C, D) = -9, u_2(C, C) = -6$

TIPS. 若有第三个参与人，则引入不同的表代表 player3 的不同选择. player 3 称为 Matrix player (矩阵参与人).
若 >3人，则不用矩阵表示.

- The payoffs of two players in Example 1 can be represented in the following bi-matrix:

		Prisoner 2	
		Defect	Confess
Prisoner 1	Defect	$-1, -1$	$-9, 0$
	Confess	$0, -9$	$-6, -6$

- Prisoner 1 is also called the row player, and Prisoner 2 the column player.
- Each entry of the bi-matrix has two numbers: the first number is the payoff of the row player and the second is that of the column player.

- Concept of Strategies
  - Important concepts:
    - Best response
    - (Strictly) dominated strategy
    - (Strictly) dominant strategy
  - Notations:

$$\begin{aligned} s &= (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \\ s_{-i} &= (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \\ S &= S_1 \times \ldots S_{i-1} \times S_i \times S_{i+1} \times \ldots S_n \\ S_{-i} &= S_1 \times \ldots S_{i-1} \times S_{i+1} \times \ldots S_n \end{aligned}$$

- Def.

In a normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, the **best response** for player i to a combination of other players' strategies $s_{-i} \in S_{-i}$, denoted by $R_i(s_{-i})$, is referred to as the set of maximizers of

$$\boxed{\max_{s_i \in S_i} u_i(s_i, s_{-i}).}$$

TIPS: 两点限制: 决策人 / 其它人的选择.

- Remark: $R_i(s_{-i}) \subset S_i$ can be an empty set, a singleton (单元素集), a finite set or an infinite set. We call R_i the **best-response correspondence** for player i.

- Def.

In a normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, let $s_i', s_i'' \in S_i$. Strategy s_i' is **strictly dominated** by strategy s_i'' (or strategy s_i'' strictly dominates strategy s_i'), if for each feasible combination of the other players' strategies, player i's payoff from playing s_i' is strictly less than player i's payoff from playing s_i'', i.e.,

$$u_i(s_i', s_{-i}) < u_i(s_i'', s_{-i}), \quad \forall s_{-i} \in S_{-i}.$$

We say s_i' is a **strictly dominated strategy** of player i.

☆: 严格被占优 不用所有的都比 s_i' 好, 存在1个 s_i'' 优于 s_i' 即可剔除 s_i'.

⇒ a rational player will never choose a strictly dominated strategy.

- Def.

In a normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, strategy $\tilde{s}_i \in S_i$ is a **strictly dominant strategy** of player i, if it strictly dominates any other strategies. Equivalently, if for each feasible combination of the other players' strategies, player i's payoff from playing $\tilde{s}_i$ is strictly larger than player i's payoff from playing any other strategies, i.e.,

$$u_i(\tilde{s}_i, s_{-i}) > u_i(\hat{s}_i, s_{-i}), \quad \forall s_{-i} \in S_{-i}, \forall \hat{s}_i \in S_i, \hat{s}_i \neq \tilde{s}_i.$$

⇒ a rational player will always choose a strictly dominant strategy.
A strictly dominant strategy is unique if it exists.

- **Result 1**: A strictly dominated strategy can never be a best response, i.e., if s_i' is a strictly dominated strategy of player i, then $s_i' \notin R_i(s_{-i})$ for all $s_{-i} \in S_{-i}$.
- **Result 2**: A strictly dominant strategy is always a best response, i.e., if $\tilde{s}_i$ is a strictly dominant strategy of player i, then $\tilde{s}_i \in R_i(s_{-i})$ for all $s_{-i} \in S_{-i}$.

重复剔除严格劣策略.

- IESDS (Iterated Elimination of Strictly Dominated Strategies).

e.g.

		Player 2		
		L	M	R
Player 1	U	$1, 0$	$1, 2$	$0, 1$
	D	$0, 3$	$0, 1$	$2, 0$

- Step 1:
  - Player 1 does not have a strictly dominated strategy.
  - For Player 2, R is a strictly dominated strategy, which is strictly dominated by M. Hence player 2 will never choose R if he is rational.
  - If player 1 knows that player 2 is rational, then he can eliminate R from player 2's strategy space by playing the following game:

		Player 2	
		L	M
Player 1	U	$1, 0$	$1, 2$
	D	$0, 3$	$0, 1$

- Step 2:
  - Now player 1 has a strictly dominated strategy, which is strategy D.
  - If player 2 also knows that i) player 1 knows that player 2 is rational, and ii) player 1 is rational, then he can also eliminate D.
  - The game is further reduced to

		Player 2	
		L	M
Player 1	U	$1, 0$	$1, 2$

- Step 3:
  - Again L is eliminated if player 1 knows that i) player 2 knows that player 1 knows that player 2 is rational, ii) player 2 knows that player 1 is rational, iii) player 2 is rational.
  - (U, M) is the final outcome!

		Player 2
		M
Player 1	U	$1, 2$

- 2 main drawbacks :
  - A key assumption: rationality of all players is **common knowledge**.
  - The prediction of IESDS may not be very precise, and sometimes it predicts nothing about games.

Notice: 一般只有理性人的假设.
这里"知道其它人是理性的"的 common knowledge 是较强的假设.

PS. 在NE中. 不需要这个假定.

e.g. IESDS can do nothing with the following game:

		Player 2		
		L	C	R
	U	0, 4	4, 0	5, 3
Player 1	M	4, 0	0, 4	5, 3
	D	3, 5	3, 5	6, 6

Lec 3. Nash Equilibrium. (NE).

所有人的1个策略组合.

Def. In the n-player normal-form game $G = \{S_1, \dots, S_n; u_1, \dots, u_n\}$, the strategies $(s_1^*, \dots, s_n^*)$ are a **Nash equilibrium** if,

$$s_i^* \in R_i(s_{-i}^*), \quad \forall i = 1, \dots, n.$$

Equivalently,

$$u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, s_{-i}^*), \quad \forall i = 1, \dots, n.$$

互为最优反应.

Then s_i^* is the equilibrium strategy of player i.

- Each player's strategy must be a best response, given other players' equilibrium strategies.
- No single player wants to deviate unilaterally → strategically stable or self-enforcing

偏离　单方面地

☆ TIPS: NE是给定他人选择的最优反应.
e.g. 在金发女郎game中. NE是3人里1人金而非4/金 1/4黑.
(pure strategy)

缺陷: 本身关注的是静态博弈. 但其实现要求每个人能预测其它人的最优选择. (这与静态博弈中假设的同时行动矛盾).

优点: NE强于 IESDS { 结果比 IESDS 更少
不要求 rationality 是 common knowledge.

- How to find a NE ?
  - For a bi-matrix game, underline the payoff to each player's best response for any given other players' strategies.
  - If you find all payoffs in a single entry are underlined, then this is a Nash equilibrium.

e.g.

		Player 2		
		L	C	R
	U	0, $\underline{4}$	$\underline{4}$, 0	5, 3
Player 1	M	$\underline{4}$, 0	0, $\underline{4}$	5, 3
	D	3, 5	3, 5	$\underline{6}$, $\underline{6}$

There exists a unique NE: (D, R).

☆. TIPS. 策略组合要写成策略的组合. 不要写成收益组合!
(6,6)

- The relationship between NE & IESDS.

Prop. In an n-player normal-form game $G = \{S_1, \dots, S_n; u_1, \dots, u_n\}$, if the strategies $(s_1^*, \dots, s_n^*)$ are a Nash equilibrium, then they survive iterated elimination of strictly dominated strategies.

i.e {NE的结果} ⊆ {IESDS的结果}

Proof: BWOC. Suppose s_i^* is the first of the strategies $(s_1^*, \cdots, s_n^*)$ to be eliminated for being strictly dominated.
$\therefore \exists s_i''$ that has not yet been eliminated from S_i that strictly dominates s_i^*.
i.e. $u_i(s_i^*, s_{-i}) < u_i(s_i'', s_{-i})$. for all s_{-i} that have not been eliminated from other players' strategy spaces.
$\because s_i^*$ is the 1st equilibrium strategy to be eliminated, we have $u_i(s_i^*, s_{-i}^*) < u_i(s_i'', s_{-i}^*)$.
This contradicts to NE. ⇕.

Prop. Consider an n-player normal-form game $G = \{S_1, \dots, S_n; u_1, \dots, u_n\}$, which is finite. If iterated elimination of strictly dominated strategies eliminates all but the strategies $(s_1^*, \dots, s_n^*)$, then these strategies are the unique Nash equilibrium of the game.

i.e. IESDS 有唯一解 ⇒ NE有唯一解.

- *Proof.* By Proposition 1, Nash equilibrium strategies can never be eliminated in IESDS. Since $(s_1^*, \dots, s_n^*)$ are the only strategies which are not eliminated, s_i^* is thus the only possible equilibrium strategy for player i. Hence, we cannot find two different Nash equilibria.
- It remains to show that $(s_1^*, \dots, s_n^*)$ are indeed a Nash equilibrium.
- We use proof by contradiction. Suppose s_i^* is not a best response of player i to s_{-i}^*.
- Let the relevant best response be b_i (which must exist since the game is finite), i.e.,

$$\max_{s_i \in S_i} u_i(s_i, s_{-i}^*) = u_i(b_i, s_{-i}^*)$$
$$> u_i(s_i^*, s_{-i}^*).$$

But b_i must be strictly dominated by some strategy t_i at some stage of the process of iterated elimination.

- So we have

$$u_i(b_i, s_{-i}) < u_i(t_i, s_{-i})$$

for all strategies (s_{-i}) that have not been eliminated from other players' strategy spaces.

- Since s_{-i}^* have not been eliminated, we have

$$u_i(b_i, s_{-i}^*) < u_i(t_i, s_{-i}^*),$$

which contradicts the fact that b_i is a best response to s_{-i}^*. ⇕

Cournot Model of Duopoly.

Set up

- Suppose two firms (1 and 2) produce a homogeneous good, and compete in quantities.
- Let q_i be the quantity produced by firm i, where $i = 1, 2$.
- The aggregate quantity of the good is denoted by $Q = q_1 + q_2$.
- The inverse demand of the good is

$$P(Q) = \begin{cases} a - Q, & \text{if } Q < a, \\ 0, & \text{if } Q \geq a. \end{cases}$$

- The cost function of firm i is $C_i(q_i) = cq_i$, where $0 < c < a$.
- Question: How much should each firm produce?

Build the model

- We first need to translate the problem into a normal-form game.
  1. Players: the two firms
  2. Strategies: $S_i = [0, \infty)$ for $i = 1, 2$ (any q_i is a strategy of firm i)
  3. Payoffs:

$$\pi_i(q_i, q_j) = \begin{cases} q_i[a - (q_i + q_j) - c], & \text{if } q_i + q_j < a, \\ -cq_i, & \text{if } q_i + q_j \geq a. \end{cases}$$

- The pair of quantities (q_1^*, q_2^*) is a Nash equilibrium if for each firm i that q_i^* solves

$$\max_{0 \leq q_i < \infty} \pi_i(q_i, q_j^*).$$

- Equivalently,

$$q_i^* \in R_i(q_j^*),$$

where $i, j = 1, 2$ and $i \neq j$.

Solve it.

- To solve for the Nash equilibrium, we first need to find the best response correspondence of each player.
- Consider the following two cases:
- Case 1: When $q_j > a - c$, player i's payoff is

$$\pi_i(q_i, q_j) \begin{cases} < 0, & \text{if } q_i > 0, \\ = 0, & \text{if } q_i = 0, \end{cases}$$

which is clearly maximized at $q_i = 0$. Thus, the best response of firm i is $R_i(q_j) = 0$.

- Case 2: When $0 \leq q_j \leq a - c$, player i's payoff is

$$\pi_i(q_i, q_j) \begin{cases} < 0, & \text{if } q_i > a - c - q_j, \\ = q_i[a - (q_i + q_j) - c], & \text{if } q_i \leq a - c - q_j. \end{cases}$$

The optimal q_i is determined by the following first-order condition

$$a - q_j - c - 2q_i = 0.$$

- Thus, the best response is $R_i(q_j) = \frac{1}{2}(a - q_j - c)$.
- In sum, the best response correspondence (or function) of player i is

$$R_i(q_j) = \begin{cases} \frac{1}{2}(a - q_j - c), & \text{if } 0 \leq q_j \leq a - c, \\ 0, & \text{if } q_j > a - c. \end{cases}$$

- The Nash equilibrium (q_1^*, q_2^*) is the intersection of two best response correspondences, which imply that

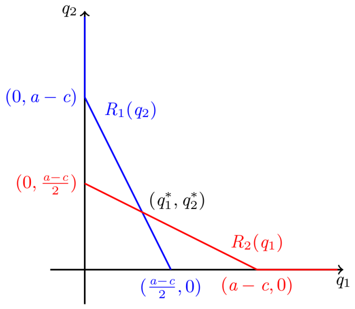
$$q_1^* = R_1(q_2^*) \text{ and } q_2^* = R_2(q_1^*).$$

- We can obtain (q_1^*, q_2^*) by simultaneously solving

$$\begin{aligned} q_1^* &= \frac{1}{2}(a - q_2^* - c), \\ q_2^* &= \frac{1}{2}(a - q_1^* - c). \end{aligned}$$

- The unique Nash equilibrium is $(q_1^*, q_2^*) = \left(\frac{a-c}{3}, \frac{a-c}{3}\right)$.

- Alternatively, we can solve for the Nash equilibrium graphically, i.e., (q_1^*, q_2^*) can be determined by the intersection of the two best response curves.



法2：用IESDS (见后).

Bertrand Model of Duopoly

Set up

- Suppose two firms produce differentiated products and compete in prices.
- The demand for firm i is

$$q_i(p_i, p_j) = a - p_i + bp_j,$$

where $b > 0$, which suggests that the two products are substitutes.

- Firms' marginal cost is again assumed to be c, where $0 < c < a$.
- Question: What is the Nash equilibrium?

Build the model

- The strategy space of firm i is $S_i = [0, \infty)$ and any $p_i \in S_i$ is a strategy.
- The profit of firm i is

$$\pi_i(p_i, p_j) = (a - p_i + bp_j)(p_i - c).$$

- The pair of prices (p_i^*, p_j^*) is a Nash equilibrium if p_i^* solves

$$\max_{0 \le p_i < \infty} (a - p_i + bp_j^*)(p_i - c),$$

which leads to

$$\big\downarrow \text{FOC}$$

$$p_i^* = \frac{1}{2}(a + bp_j^* + c).$$

Solve it.

- The Nash equilibrium is determined by

$$\begin{aligned} p_1^* &= \frac{1}{2}(a + bp_2^* + c), \\ p_2^* &= \frac{1}{2}(a + bp_1^* + c). \end{aligned}$$

- The unique Nash equilibrium is $(p_1^*, p_2^*) = \left(\frac{a+c}{2-b}, \frac{a+c}{2-b}\right)$.
- The problem only makes sense if $b < 2$.

The Problem of the Commons

Set up.

- Suppose n farmers graze their goats on the village green.
- The number of goats that the i^{th} farmer owns is g_i and the total number of goats in the village is denoted by $G = g_1 + \cdots g_n$.
- The cost of buying and caring for a goat is c.
- The value to a farmer is $v(G)$ per goat.
  - Maximum number of goats that can be grazed is G_{max}, where $v(G) > 0$ for $G < G_{max}$ and $v(G) = 0$ for $G \ge G_{max}$.
  - For $G < G_{max}$, $v'(G) < 0$ and $v''(G) < 0$.
- Assume that goats are continuously divisible and farmers simultaneously choose how many goats to graze.
- Question: What should farmers do? Are their choices socially optimal?

Build the model.

- The normal-form representation of the game:
  1. Players: n farmers
  2. Strategies: $S_i = [0, G_{max})$ (g_i is a strategy of farmer i)
  3. Payoffs:

$$u_i(g) = g_i v(g_i + g_{-i}) - cg_i,$$

where $g = (g_1, \cdots, g_n)$ and $g_{-i} = G - g_i$

- If $(g_1^*, \cdots, g_n^*)$ are a Nash equilibrium, then g_i^* must solve

$$\text{FOC} \quad v(g_i^* + g_{-i}^*) + g_i^* v'(g_i^* + g_{-i}^*) - c = 0.$$

- Summing up all n first-order conditions yields

$$v(G^*) + \frac{1}{n} G^* v'(G^*) - c = 0$$

for $G^* = g_1^* + \cdots + g_n^*$.

What's the problem?

- The social optimum (denoted by G^{**}) solves

(social welfare maximize)

$$\max_{0 \le G < \infty} Gv(G) - Gc,$$

which is given by

$$v(G^{**}) + G^{**} v'(G^{**}) - c = 0.$$

- Comparing G^{**} with G^*, we have $G^* > G^{**}$:
  - Too many goats are grazed in the Nash equilibrium, compared to the social optimum.
  - The common resource is overutilized because each farmer considers his or her own incentives, but not other farmers'.

Lec 4. Mixed Strategies

Def. In a normal-form game $G = \{S_1, \dots, S_n; u_1, \dots, u_n\}$, suppose $S_i = \{s_{i1}, \dots, s_{iK_i}\}$. Each strategy $s_{ik} \in S_i$ is a **pure strategy** for player i. A **mixed strategy** for player i is a probability distribution $p_i = (p_{i1}, \dots, p_{iK_i})$, for $k = 1, \dots, K_i$, where $p_{i1} + \cdots + p_{iK_i} = 1$ and $p_{ik} \geq 0$.

参考理解：Sik是一组选择，参与者可以用任意比例组合它们。
类似方程的基解。

- In the Matching Pennies example, $S_i = \{\text{Heads, Tails}\}$.
- Each player has two pure strategies: Heads or Tails.
- A mixed strategy for a player is a probability distribution $(p, 1-p)$, where p is the probability that the player chooses Heads, while $1-p$ is the probability that the player chooses Tails.
- $(1/2, 1/2)$ means playing Heads and Tails with an equal probability; $(1/3, 2/3)$ means playing Heads with a probability of $1/3$ and Tails with a probability of $2/3$.
- The mixed strategy $(1, 0)$ is simply a pure strategy of playing Heads.

		Player 2	
		Heads	Tails
Player 1	Heads	$-1, 1$	$1, -1$
	Tails	$1, -1$	$-1, 1$

- How to extend the definition of Nash equilibrium to include mixed strategies?
- Consider the case with two players.
- Suppose
$$S_1 = \{s_{11}, s_{12}, \dots, s_{1J}\},$$
and
$$S_2 = \{s_{21}, s_{22}, \dots, s_{2K}\}.$$
- Each $s_{1j} \in S_1$ is a pure strategy for player 1, and each $s_{2k} \in S_2$ is a pure strategy for player 2.
- If player 1 thinks that player 2 will play a mixed strategy $p_2 = (p_{21}, \dots, p_{2K})$, then player 1's expected payoff of playing a pure strategy s_{1j} is
$$v_1(s_{1j}, p_2) = \sum_{k=1}^{K} p_{2k} u_1(s_{1j}, s_{2k}).$$
- Player 1's expected payoff of playing a mixed strategy $p_1 = (p_{11}, \dots, p_{1J})$ is
$$\begin{aligned} v_1(p_1, p_2) &= \sum_{j=1}^{J} p_{1j} \sum_{k=1}^{K} p_{2k} u_1(s_{1j}, s_{2k}) \\ &= \sum_{j=1}^{J} \sum_{k=1}^{K} p_{1j} p_{2k} u_1(s_{1j}, s_{2k}). \end{aligned}$$
- A mixed strategy $p_1 = (p_{11}, \dots, p_{1J})$ is a best response to p_2 if
$$v_1(p_1, p_2) \geq v_1(p_1', p_2),$$
for all p_1' over S_1.
- Similarly, if player 2 thinks player 1 will play a mixed strategy $p_1 = (p_{11}, \dots, p_{1J})$, then player 2's expected payoff of playing a mixed strategy $p_2 = (p_{21}, \dots, p_{2K})$ is
$$\begin{aligned} v_2(p_1, p_2) &= \sum_{K=1}^{K} p_{2k} \sum_{j=1}^{J} p_{1j} u_2(s_{1j}, s_{2k}) \\ &= \sum_{j=1}^{J} \sum_{k=1}^{K} p_{1j} p_{2k} u_2(s_{1j}, s_{2k}). \end{aligned}$$

• Mixed-Strategy Nash Equilibrium

Def. In a two-player normal-form game $G = \{S_1, S_2; u_1, u_2\}$, the mixed strategies (p_1^*, p_2^*) are a **Nash equilibrium** if each player's mixed strategy is a best response to the other player's mixed strategy:
$$v_1(p_1^*, p_2^*) \geq v_1(p_1, p_2^*) \text{ for every } p_1 \text{ over } S_1,$$
and
$$v_2(p_1^*, p_2^*) \geq v_2(p_1^*, p_2) \text{ for every } p_2 \text{ over } S_2.$$

★TIPS. 考试时注意看问的是纯策略还是混合策略的NE.

- How to find mixed-strategy NE?

- We consider the case with two players, each having two pure strategies.
- Let $p_1 = (r, 1-r)$ be a mixed strategy for player 1 and $p_2 = (q, 1-q)$ be a mixed strategy for player 2.
- Player 1's expected payoff of playing p_1, given player 2's strategy p_2, is

对player 1来讲是常数.

$$v_1(p_1, p_2) \quad = \quad r\underline{v_1(s_{11}, p_2)} + (1-r)\underline{v_1(s_{12}, p_2)}.$$

- For each p_2 (or q), we need to compute r, denoted by $r^*(q)$, such that p_1 is a best response to p_2.
- $r^*(q)$ is the set of solutions to

$$\max_r v_1(p_1, p_2),$$

where

$$r^*(q) = \begin{cases} 1, & \text{if } v_1(s_{11}, p_2) > v_1(s_{12}, p_2); \\ [0,1], & \text{if } v_1(s_{11}, p_2) = v_1(s_{12}, p_2); \\ 0, & \text{if } v_1(s_{11}, p_2) < v_1(s_{12}, p_2). \end{cases}$$

- Similarly, player 2's expected payoff is

$$v_2(p_1, p_2) = qv_2(p_1, s_{21}) + (1-q)v_2(p_1, s_{22}).$$

- Given p_1, the best response for player 2 is denoted by $q^*(r)$, which is the set of solutions to

$$\max_q v_2(p_1, p_2),$$

where

$$q^*(r) = \begin{cases} 1, & \text{if } v_2(p_1, s_{21}) > v_2(p_1, s_{22}); \\ [0,1], & \text{if } v_2(p_1, s_{21}) = v_2(p_1, s_{22}); \\ 0, & \text{if } v_2(p_1, s_{21}) < v_2(p_1, s_{22}). \end{cases}$$

- A mixed strategy Nash equilibrium is an intersection of the two best-response correspondences $r^*(q)$ and $q^*(r)$.
- If (r^*, q^*) is a mixed strategy Nash equilibrium, then

$$\begin{aligned} r^* &= r^*(q^*), \\ q^* &= q^*(r^*). \end{aligned}$$

e.g.

- Find a Nash equilibrium in the game of Matching Pennies.

		Player 2	
		Heads q	Tails $1-q$
Player 1	Heads r	$-1, 1$	$1, -1$
	Tails $1-r$	$1, -1$	$-1, 1$

- Let $p_1 = (r, 1-r)$ be a mixed strategy for player 1, where r is the probability player 1 chooses Heads.

TIPS. 要写清r是什么的概率.

- Similarly, let $p_2 = (q, 1-q)$ be a mixed strategy for player 2, where q is the probability player 2 chooses Heads.
- What is $r^*(q)$ and $q^*(r)$?
- For player 1,

$$\begin{aligned} v_1(s_{11}, p_2) &= q \cdot (-1) + (1-q) \cdot 1 = 1 - 2q, \\ v_1(s_{12}, p_2) &= q \cdot 1 + (1-q) \cdot (-1) = -1 + 2q. \end{aligned}$$

- Player 1 chooses Heads (i.e., $r^*(q) = 1$) if and only if

$$1 - 2q > -1 + 2q \Leftrightarrow 0 \le q < 1/2.$$

- We have

$$r^*(q) = \begin{cases} 1, & \text{if } 0 \le q < 1/2; \\ [0,1], & \text{if } q = 1/2; \\ 0, & \text{if } 1/2 < q \le 1. \end{cases}$$

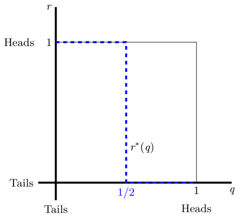


Figure 1: Best response correspondence for player 1: $r^*(q)$

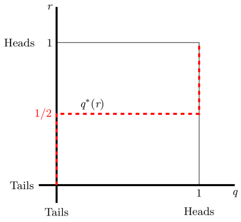


Figure 2: Best response correspondence for player 2: $q^*(r)$

- For player 2,

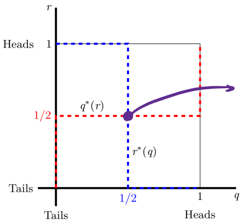
$$\begin{aligned} v_2(p_1, s_{21}) &= r \cdot 1 + (1-r) \cdot (-1) = -1 + 2r, \\ v_2(p_1, s_{22}) &= r \cdot (-1) + (1-r) \cdot 1 = 1 - 2r. \end{aligned}$$

- Player 2 chooses Heads (i.e., $q^*(r) = 1$) if and only if

$$-1 + 2r > 1 - 2r \Leftrightarrow 1/2 < r \le 1.$$

- We have

$$q^*(r) = \begin{cases} 1, & \text{if } 1/2 < r \le 1; \\ [0,1], & \text{if } r = 1/2; \\ 0, & \text{if } 0 \le r < 1/2. \end{cases}$$



$P_1^* = (\frac{1}{2}, \frac{1}{2})$.
$P_2^* = (\frac{1}{2}, \frac{1}{2})$.

the only NE in mixed strategies.

Figure 3: Mixed-strategy Nash equilibrium in Matching Pennies

Prop. The pure strategies played with a positive probability in a mixed-strategy Nash equilibrium survive IESDS.

i.e. NE强于IESDS.

用处：3个及以上纯策略组成混合策略时求NE. 可以先IESDS去掉一些策略.

• General case:

- In general, let $p = (p_1, \dots, p_n)$ be a mixed strategy profile, where $p_i = (p_{i1}, \dots, p_{iK_i})$, for $i = 1, \dots, n$.
- The expected payoff for player i is

$$v_i(p) = \sum_{j=1}^{K_i} p_{ij} v_i(p_1, \dots, p_{i-1}, s_{ij}, p_{i+1}, \dots, p_n).$$

- The mixed strategy p_i^* is a best response to $p_{-i} = (p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ if

$$v_i(p_i^*, p_{-i}) \geq v_i(p_i, p_{-i})$$

for all probability distribution p_i over S_i.

Def. In a normal-form game $G = \{S_1, \dots, S_n; u_1, \dots, u_n\}$, the mixed strategies $(p_1^*, \dots, p_n^*)$ are a **(mixed-strategy) Nash equilibrium** if each player's mixed strategy is a best response to the other players' mixed strategies in terms of expected payoff, i.e.,

$$v_i(p_i^*, p_{-i}^*) \geq v_i(p_i, p_{-i}^*)$$

for every p_i over S_i, and for all $i = 1, \dots, n$.

• Existence of NE.

Thm. In the n-player normal-form game $G = \{S_1, \dots, S_n; u_1, \dots, u_n\}$, if n is finite and S_i is finite for every i, then there exists at least one Nash equilibrium, possibly involving mixed strategies.

ps. 这是NE存在的充分不必要条件.

• Strictly Dominated Strategy and Best Response.

- Before we know that if a (pure) strategy is a strictly dominated strategy, then it can never be a best response.
- But the reverse may not be true.
- Once we have considered mixed strategies, then the reverse can also be true.
- For instance, in a two-player game, a pure strategy is a strictly dominated strategy if and only if it is never a best response.

- A pure strategy can be strictly dominated by a mixed strategy, even if it is not strictly dominated by any pure strategy!
- Example:

Player 2

		L	R
Player 1	U	3, –	0, –
	M	0, –	3, –
	D	1, –	1, –

- D is not strictly dominated by either U or M.
- But D is strictly dominated by a strategy $(1/2, 1/2, 0)$, i.e., playing U and M with a half probability.
- D is a strictly dominated strategy → D is never a best response.

- A pure strategy can be a best response to a mixed strategy, even if it is not a best response to any pure strategy!

Player 2

		L	R
Player 1	U	3, –	0, –
	M	0, –	3, –
	D	2, –	2, –

- D is not a best response to L or R.
- D is a best response to a mixed strategy $(q, 1-q)$ chosen by player 2, if

$$2 \geq 3q \text{ and } 2 \geq 3(1-q),$$

i.e., $1/3 \leq q \leq 2/3$.

- D is not a "never best response" → D is not a strictly dominated strategy!

Chapter 2 Dynamic Games

Lec 5.

• Dynamic Games of Complete Information

• Lead in example

- Consider a two-move game between two players. First, player 1 decides whether to give \$1000 to player 2. Second, after observing the choice of player 1, player 2 chooses whether to explode a grenade that will kill both of them. Player 2 can threaten player 1 by saying "Give the money to me, otherwise I will explode the grenade to kill you!"
- Question: What should player 1 do in the first place? Is player 2's threat credible to player 1? What is the outcome of this simple game?

- On a winter evening, a farmer found a snake frozen with cold. The farmer wanted to save the snake, which would make himself happy. But he was worried if the snake would bite him after it was saved. Believing that the snake would be grateful, the farmer saved it. However, when the snake was recovered, it bit and killed the farmer immediately.
- Question: Why shouldn't the farmer save the snake?

• Introduction

- These are examples of **dynamic games**.
- The central issue of dynamic games is **credibility**.
- Dynamic: sequential choice, or repeated play
- Complete information: each player's payoff function is common knowledge among all players.

- Two types of dynamic games of complete information:
  1. Dynamic games of complete and **perfect information**
  2. Dynamic games of complete and **imperfect information**
- In static games of complete information, we use **normal-form representation** to describe a game.
- Now we use **extensive-form representation** for dynamic games.
- In particular, we will draw **game trees**.

- Consider a two-player and two-stage game.
- Player 1 chooses an action L or R.
- Player 2 observes player 1's action and then chooses an action L' or R'.
- Each path (a combination of two actions) in the following tree is followed by two payoffs: the first for player 1 and the second for player 2.

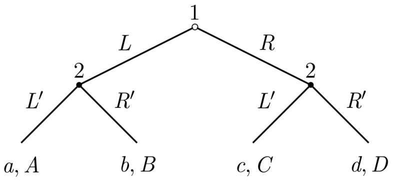


Figure 1: Extensive-form representation using a game tree

- The above game is an example of dynamic games of complete and perfect information.
- This type of games takes the following form:
  - Player 1 chooses an action a_1 from the feasible set A_1;
  - Player 2 observes a_1 and then chooses an action a_2 from the feasible set A_2;
  - Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$.
- Note that
  - A_2 may depend on the action a_1, i.e., $A_2(a_1)$.
  - Some action a_1 may even end the game, so that $A_2(a_1)$ is an empty set (i.e., no choice of player 2).

- In Example 1:
- $A_1 = \{L, R\}$, where $L =$ "give \$1000" and $R =$ "don't give";
- $A_2(L) = A_2(R) = \{L', R'\}$, where $L' =$ "explode" and $R' =$ "don't explode".

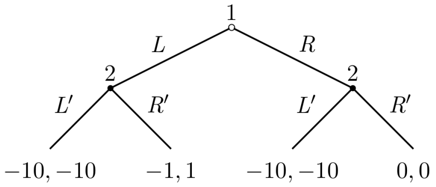


Figure 2: A game tree for Example 1

- In Example 2:
- $A_1 = \{L, R\}$, where $L =$ "save" and $R =$ "don't save";
- $A_2(L) = \{L', R'\}$, where $L' =$ "bite" and $R' =$ "don't bite";
- $A_2(R) = \emptyset$.

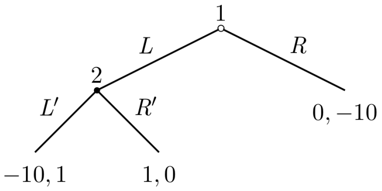


Figure 3: A game tree for Example 2

- Some key features of dynamic games of complete and perfect information:
  1. The moves occur in sequence;
  2. All previous moves are observed before the next move is chosen;
  3. The players' payoffs from each combination of moves are common knowledge.
- How to solve this type of games?
- We use **backwards induction**.

- In the second stage, player 2 observes the action (say a_1) chosen by player 1 in the first stage, and then chooses an action by solving

$$\max_{a_2 \in A_2} u_2(a_1, a_2).$$

- Assume this optimization problem has a unique solution, denoted by $R_2(a_1)$. This is player 2's best response to player 1's action a_1.
- For example, $R_2(L) = R'$ and $R_2(R) = L'$.
- In the first stage, knowing player 2's best response, player 1's problem becomes

$$\max_{a_1 \in A_1} u_1(a_1, R_2(a_1)).$$

- Assume it also has a unique solution, denoted by a_1^*.
- For example, $a_1^* = R$ and $R_2(a_1^*) = L'$. 逆向归纳解
- We call $(a_1^*, R_2(a_1^*))$ the **backwards-induction outcome** of the game.

- In Example 1:
  - $R_2(L) = R_2(R) = R'$;
  - $a_1^* = R$ and $R_2(a_1^*) = R'$;
  - The backwards-induction outcome is (R, R').
- In Example 2:
  - $R_2(L) = L'$;
  - $a_1^* = R$;
  - The backwards-induction outcome is R.

- Stackelberg Model of Duopoly

- Consider a dominant firm moving first and a follower moving second.
- The game is played as follows:
  - Firm 1 chooses a quantity $q_1 \geq 0$.
  - Firm 2 observes q_1 and then chooses a quantity $q_2 \geq 0$.
  - The payoff of firm i is the profit

$$\pi_i(q_1, q_2) = q_i[P(Q) - c],$$

where $Q = q_1 + q_2$ and

$$P(Q) = \begin{cases} a - Q, & \text{if } Q < a; \\ 0, & \text{if } Q \geq a. \end{cases}$$

- How to find the backwards-induction outcome?
- First, find the best response function $R_2(q_1)$ for firm 2, i.e., for any given q_1, find q_2 that solves

$$\max_{q_2 \geq 0} \pi_2(q_1, q_2),$$

where

$$\pi_2(q_1, q_2) = \begin{cases} q_2(a - q_1 - q_2 - c), & \text{if } q_1 + q_2 < a; \\ -cq_2, & \text{if } q_1 + q_2 \geq a. \end{cases}$$

- Then we have

$$R_2(q_1) = \begin{cases} \frac{a-c-q_1}{2}, & \text{if } q_1 < a - c; \\ 0, & \text{if } q_1 \geq a - c. \end{cases}$$

- $R_2(q_1)$ is the same as that in the Cournot model.
- Second, firm 1 knows $R_2(q_1)$ and solves

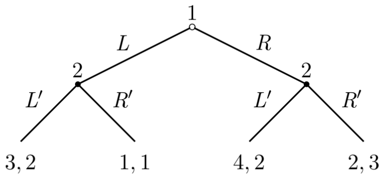
$$\max_{q_1 \geq 0} \pi_1(q_1, R_2(q_1)),$$

where

$$\pi_1(q_1, R_2(q_1)) = \begin{cases} q_1\left[a - q_1 - \frac{a-q_1-c}{2} - c\right], & \text{if } q_1 < a - c; \\ q_1(a - q_1 - c), & \text{if } a - c \leq q_1 < a; \\ -cq_1, & \text{if } q_1 \geq a. \end{cases}$$

Notice：Backwards-induction outcome 和 NE 可能不同.

- Consider the following game:



- $R_2(L) = L'$ and $R_2(R) = R'$.
- The backwards-induction outcome is (L, L').
- Suppose both players choose actions simultaneously, then they play the following game:

		Player 2	
		L'	R'
Player 1	L	3, 2	1, 1
	R	4, 2	2, 3

- The Nash equilibrium is (R, R'), which differs from the backwards-induction outcome (L, L').
- The backwards-induction outcome in a dynamic game could be different from the Nash equilibrium of the corresponding game played simultaneously.

- Clearly, for $q_1 > a - c$, firm 1's profit is always negative.
- Thus we only need to solve

$$\max_{q_1 \geq 0} q_1 \left[a - q_1 - \frac{a - q_1 - c}{2} - c \right] = \max_{q_1 \geq 0} \left[\frac{1}{2} q_1 (a - q_1 - c) \right],$$

which leads to the following first-order condition

$$a - c - 2q_1 = 0.$$

- The optimal choice of firm 1 is

$$q_1^* = \frac{a - c}{2}.$$

- The quantity chosen by firm 2 is

$$q_2^* = R_2(q_1^*) = \frac{a - c}{4}.$$

- The market price is

$$P^* = a - q_1^* - q_2^* = c + \frac{a - c}{4}.$$

- Firms' profits and the total profit are

$$\pi_1^* = \frac{(a-c)^2}{8}, \quad \pi_2^* = \frac{(a-c)^2}{16}, \text{ and } \Pi^* = \pi_1^* + \pi_2^* = \frac{3(a-c)^2}{16}.$$

- Comparison between Cournot model and Stackelberg model:

Table 1: Cournot Model vs. Stackelberg Model

Variable	Cournot Model	Stackelberg Model
q_1^*	$\frac{a-c}{3}$	$\frac{a-c}{2}$
q_2^*	$\frac{a-c}{3}$	$\frac{a-c}{4}$
π_1^*	$\frac{(a-c)^2}{9}$	$\frac{(a-c)^2}{8}$
π_2^*	$\frac{(a-c)^2}{9}$	$\frac{(a-c)^2}{16}$
Π^*	$\frac{2(a-c)^2}{9}$	$\frac{3(a-c)^2}{16}$
P^*	$c + \frac{a-c}{3}$	$c + \frac{a-c}{4}$

- Dynamic Games of Imperfect Information

  - Consider the following simple two-stage game:
    - Players 1 and 2 simultaneously choose actions a_1 and a_2 from the feasible sets A_1 and A_2, respectively.
    - Players 3 and 4 observe the outcome of the first stage (a_1, a_2) and then simultaneously choose actions a_3 and a_4 from the feasible sets A_3 and A_4, respectively.
    - Payoffs are $u_i(a_1, a_2, a_3, a_4)$ for $i = 1, 2, 3, 4$.
  - This game differs from the two-stage game with perfect information, since there are simultaneous moves within each stage.
  - We solve this game by using the idea of backwards induction.
  - For each given (a_1, a_2), players 3 and 4 try to find the Nash equilibrium in stage 2.
  - Assume the second-stage game has a unique Nash equilibrium

$$(a_3^*(a_1, a_2), a_4^*(a_1, a_2)).$$

先对所有outcome求NE. 再转化成 complete information 求解.
(同时进行的game的)

  - Then, player 1 and player 2 play a simultaneous-move game with payoffs

$$u_i(a_1, a_2, a_3^*(a_1, a_2), a_4^*(a_1, a_2)), \text{ for } i = 1, 2.$$

  - Suppose (a_1^*, a_2^*) is the unique Nash equilibrium of this simultaneous-move game.
  - Then

$$(a_1^*, a_2^*, a_3^*(a_1^*, a_2^*), a_4^*(a_1^*, a_2^*))$$

is the **subgame-perfect outcome** of the two-stage game.

- e.g. Bank Runs 银行挤兑博弈.

  - Two investors have each deposited \$5 millions with a bank. The bank has invested these deposits in a long-term project.
  - If the bank is forced to liquidate its investment before the project matures, a total of \$8 millions can be recovered.
  - If the bank allows the investment to reach maturity, the project will pay out a total of \$16 millions.
  - There are two dates at which the investors can make withdrawals at the bank: Date 1 is before the bank's investment matures and Date 2 is after.
  - Suppose there is no discounting.
  - We work backwards.
  - At date 2, in the unique Nash equilibrium, both withdraw and each obtains \$8 millions.
  - At date 1, they play the following game:

	Withdraw	Don't
Withdraw	4, 4	5, 3
Don't	3, 5	8, 8

  - There are 2 pure-strategy Nash equilibria of this game:
    1. Both withdraw and each obtains \$4 millions;
    2. Both don't and each obtains \$8 millions.

  - Players' payoffs in date 1:

	Withdraw	Don't
Withdraw	4, 4	5, 3
Don't	3, 5	next stage

  - Players' payoffs in date 2:

	Withdraw	Don't
Withdraw	8, 8	11, 5
Don't	5, 11	8, 8

- There are 2 subgame-perfect outcomes of the original two-stage game:
  1. Both withdraw at date 1 to obtain \$4 millions each → the case of bank run
  2. Both don't withdraw at date 1 but do at date 2, and each obtains \$8 millions.
- Although there are two possible subgame-perfect outcomes, only the second one is efficient.
- This model does not predict when bank runs will occur, but does show that they can occur as an equilibrium outcome.

Lec 6. Extensive-Form Representation of Games and Subgame-Perfect NE.

• Normal-Form Representation of Games

Def.

The **normal-form representation** of a game specifies
(1) the players in the game;
(2) the strategies available to each player;
(3) the payoff received by each player for each combination of strategies that could be chosen by the players.

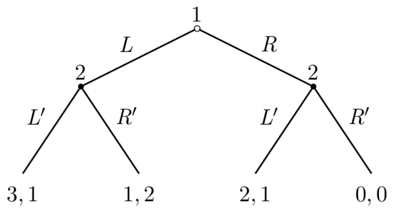
→ Strategies VS. moves.

• Extensive-Form Representation of Games

Def.

The **extensive-form representation** of a game specifies:
(1) the players in the game;
(2a) when each player has the move;
(2b) what each player can do at each of his or her opportunities to move;
(2c) what each player knows at each of his or her opportunities to move;
(3) the payoffs received by each player for each combination of moves that could be chosen by the players.

- Example 1:



- In Example 1, the game tree begins with a **decision node** 节点 for player 1, which is also the **initial node** of the game.
- After player 1's choice (L or R) is made, player 2's decision node is reached. And player 2 needs to decide whether to choose L' or R'.
- A **terminal node** is reached after player 2's move (i.e., the game ends), and payoffs of players are realized.

TIPS. 每个 decision node 上要标注决策人!

• Information Set.

- For games with imperfect information, some previous moves are not observed by the player with the current move.
- To present this kind of ignorance of previous moves and to describe what each player knows at each of his/her move, we introduce the notion of a player's **information set**.

Def.

An **information set** for a player is a collection of decision nodes satisfying:
(i) The player needs to move at every node in the information set.
(ii) When the play of the game reaches a node in the information set, the player with the move does not know which node in the set has (or has not) been reached.

- (ii) implies that the player must have the same set of feasible actions at each decision node in an information set, otherwise the player could infer from the set of actions available that some node(s) had or had not been reached.

- In an extensive-form game, a collection of decision nodes, which constitutes an information set, is connected by a dotted line.
- We can use information set to differentiate perfect and imperfect information.
- A game is of **perfect information** if every information set is a singleton, and of **imperfect information** if there is at least one non-singleton information set.

- Let's consider a two-player simultaneous-move (static) game as follows:
  1. Player 1 chooses $a_1 \in A_1$;
  2. Player 2 does not observe player 1's move but chooses an $a_2 \in A_2$;
  3. Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$.
- We need an information set to describe player 2's ignorance of player 1's actions.
- The above static game of complete information can be represented as a dynamic game of complete but imperfect information.

- Strategy

Def. A **strategy** for a player is a complete plan of actions. It specifies a feasible action for the player in every contingency in which the player might be called on to act.

可能发生的事

- An equivalent definition: A player's **strategy** is a function which assigns an action to each information set (*not* each decision node) belonging to the player.
- An action and a strategy do not make a big difference in static games, while they do in dynamic games.

- Subgame - Perfect NE.

先不管信息集正常找NE.
再去掉非子博弈精炼的.

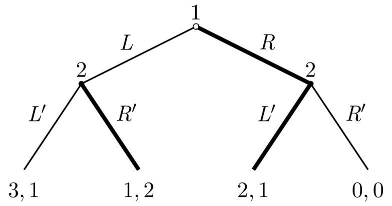
Def. A **subgame** in an extensive-form game
(a) begins at a decision node n that is a singleton information set (but is not the game's initial node);
(b) includes all the decision and terminal nodes following node n in the game tree (but no nodes that do not follow n);
(c) does not cut any information sets (i.e., if a decision node n' follows n in the game tree, then all other nodes in the information set containing n' must also follow n, and so must be included in the subgame).

Def. A Nash equilibrium is **subgame-perfect** if the players' strategies constitute a Nash equilibrium in every subgame.

即. 总体NE + 子博弈NE双重验证.

Prop. finite dynamic game of complete information 有 subgame-perfect NE. (有可能是混合策略).

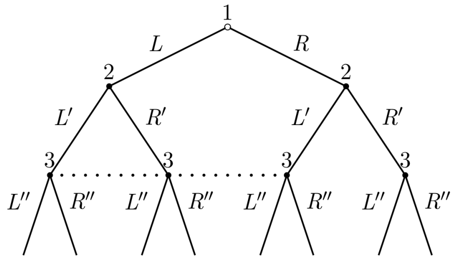
- In Example 1, there are two subgames: in the left subgame, the Nash equilibrium involves the player 2 choosing R'; in the right subgame, the Nash equilibrium involves the player 2 choosing L'.
- The subgame-perfect Nash equilibrium is $(R, (R', L'))$.
- We can use thick lines to represent the equilibrium paths.



- Subgame-perfect Nash equilibrium is closely related to two previous concepts:
  1. backwards-induction outcome
  2. subgame-perfect outcome
- What's the difference between an equilibrium and an outcome?
- An equilibrium is a collection of players' strategy profiles, while an outcome is a collection of players' actions.

- Consider the following two-stage game of complete and perfect information:
  1. Player 1 chooses an action $a_1 \in A_1$;
  2. Player 2 observes a_1 and then chooses an action $a_2 \in A_2$;
  3. Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$.
- The best response $R_2(a_1)$ solves $\max_{a_2 \in A_2} u_2(a_1, a_2)$.
- a_1^* solves $\max_{a_1 \in A_1} u_1(a_1, R_2(a_1))$.
- The backwards-induction outcome is $(a_1^*, R_2(a_1^*))$.
- The subgame-perfect Nash equilibrium is $(a_1^*, R_2(\cdot))$.
- Note that $R_2(a_1^*)$ is an action, while $R_2(\cdot)$ is a strategy for player 2.

- Example 3:
- Player 3 has a non-singleton information set and a singleton information set.



⇒ 若3在虚线处. 则只知道他在三个中的任意一个.
若3在最右边. 则知道他在最右边 (1:R. 2:R').

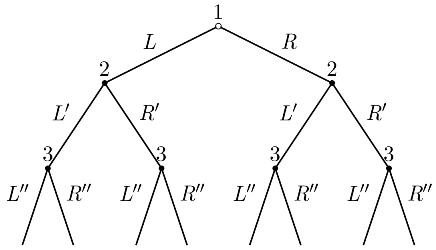
- In Example 3:
- Player 1 has two strategies: L and R.
- Player 2 has four strategies:

$$(L', L'); (L', R'); (R', L'); (R', R').$$

- Player 3 has four strategies

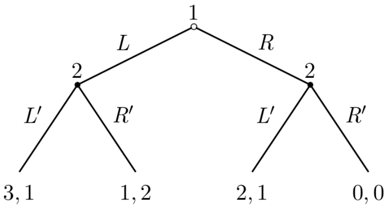
$$(L'', L''); (L'', R''); (R'', L''); (R'', R'').$$

- Example 4:
- Player 3 has 4 singleton information sets.



- Player 3 has 16 strategies.
- For instance, the strategy $\boxed{(L'', R'', R'', L'')}$ means:
  - if player 1 plays L and player 2 plays L', then player 3 plays L'';
  - if player 1 plays L and player 2 plays R', then player 3 plays R''
  - if player 1 plays R and player 2 plays L', then player 3 plays R''
  - if player 1 plays R and player 2 plays R', then player 3 plays L''

- Example 1:



- In Example 1:
- (R, L') is the backwards-induction outcome, while $(R, (R', L'))$ is the subgame-perfect Nash equilibrium.

- In the Stackelberg model:
- The backwards-induction outcome is (q_1^*, q_2^*), where $q_1^* = \frac{a-c}{2}$ and $q_2^* = \frac{a-c}{4}$, while the subgame-perfect Nash equilibrium is $(q_1^*, R_2(q_1))$, where $R_2(q_1) = \begin{cases} \frac{a-c-q_1}{2}, & q_1 < a-c \\ 0, & q_1 \geq a-c \end{cases}$.

- Consider the following two-stage game of complete but imperfect information:
  - Players 1 and 2 simultaneously choose actions a_1 and a_2 from the feasible sets A_1 and A_2, respectively.
  - Players 3 and 4 observe the outcome of the first stage (a_1, a_2) and then simultaneously choose actions a_3 and a_4 from the feasible sets A_3 and A_4, respectively.
  - Payoffs are $u_i(a_1, a_2, a_3, a_4)$ for $i = 1, 2, 3, 4$.
- For each given (a_1, a_2), players 3 and 4 play the Nash equilibrium in stage 2

$$(a_3^*(a_1, a_2), a_4^*(a_1, a_2)).$$

- Then, player 1 and player 2 play a simultaneous-move game with payoffs

$$u_i(a_1, a_2, a_3^*(a_1, a_2), a_4^*(a_1, a_2)), i = 1, 2$$

- Suppose (a_1^*, a_2^*) is the unique Nash equilibrium in stage 1.
- Then the subgame-perfect outcome is

$$(a_1^*, a_2^*, a_3^*(a_1^*, a_2^*), a_4^*(a_1^*, a_2^*)).$$

- The subgame-perfect Nash equilibrium is

$$(a_1^*, a_2^*, a_3^*(a_1, a_2), a_4^*(a_1, a_2)).$$

- NE v.s. Subgame-Perfect NE.

- A Nash equilibrium may not be subgame-perfect.
- In Example 1, the normal-form representation is

		Player 2			
		(L', L')	(L', R')	(R', L')	(R', R')
Player 1	L	3, 1	3, 1	1, 2	1, 2
	R	2, 1	0, 0	2, 1	0, 0

- Two Nash equilibria: $(L, (R', R'))$ and $(R, (R', L'))$
- Only one subgame-perfect Nash equilibrium: $(R, (R', L'))$
- The Nash equilibrium $(R, (R', L'))$ is subgame-perfect, because R' and L' are the optimal strategies in the left and right subgames, respectively, where player 2 is the only player.
- On the other hand, the Nash equilibrium $(L, (R', R'))$ is not subgame-perfect, because when player 1 chooses R, R' is not optimal to player 2 in the right subgame, i.e., R' is not a Nash equilibrium in that subgame.
- One can think the strategy (R', R') by player 2 as a threat to player 1.
- Nash equilibria that rely on non-credible threats or promises can be eliminated by the requirement of subgame perfection.
- Subgame-perfect Nash equilibrium is a refinement of Nash equilibrium, i.e.,

精炼.

$$\{\text{Subgame-perfect Nash equilibria}\} \subset \{\text{Nash equilibria}\}$$

Lec 7、8 Repeated Games

重复多次简单静态博弈.

• Introduction

- In a long-term relationship, one must consider how his/her current behavior will influence others' behavior in the future, or how threats or promises about future behavior can affect current behavior.
- In these dynamic situations, one might care about "reputation", which is often used to describe how a person's past actions affect future beliefs and behavior.

- We use **repeated games** to study such interactions among players.
- In repeated games, we are interested in how repeated interactions among players would affect their behavior.
- Two types of repeated games:
  - finitely repeated games
  - infinitely repeated games

• Finitely Repeated Games

• 2-stage Prisoners' Dilemma

等同于上一Lec的两阶段不完美信息博弈.

- The two players play the simultaneous-move game twice;
- Each player observes the outcome of the first play before the second game begins;

(否则重复两期没有差别)

- The payoff of each player in the whole game is simply the sum of two payoffs in both stages (i.e., no discounting).

- We can use backwards induction to solve the game.
- In stage 2, the unique Nash equilibrium is (L_1, L_2), in which each player receives 1.
- In stage 1, the two players play the following equivalent game:

Player 2

		L_2	R_2
Player 1	L_1	$2,2$	$6,1$
	R_1	$1,6$	$5,5$

- Hence, (L_1, L_2) is the unique Nash equilibrium in stage 1.
- The subgame-perfect outcome: (L_1, L_2) is played in both periods.

⇒ NE: ((L1, L1, L1, L1, L1), (L2, L2, L2, L2, L2))

Def.

行动空间 (区别于策略空间)

- Let $G = \{A_1, \dots, A_n; u_1, \dots, u_n\}$ denote a static game of complete information in which players 1 through n simultaneously choose actions a_1 through a_n from the action spaces A_1 through A_n, and the payoffs are $u_1(a_1, \dots, a_n)$ through $u_n(a_1, \dots, a_n)$.
- The game G is called the **stage game** of the repeated game.

Def.

Given a stage game G, let $G(T)$ denote the **finitely repeated game** in which G is played T times, with the outcomes of all preceding plays observed before the next play begins. The payoffs for $G(T)$ are simply the sum of the payoffs from the T stage games.

Prop.

If the stage game G has a unique Nash equilibrium then, for any finite T, the repeated game $G(T)$ has a unique subgame-perfect outcome: the Nash equilibrium of G is played in every stage.

- In the Prisoners' Dilemma example, the unique outcome in each period is (L_1, L_2) regardless of how many times the game is played.

- The result in the above proposition can be extended even if G itself is a dynamic game of complete information.

• What if the stage game G has multiple NE?

⇒ Then there may be subgame-perfect outcomes of the repeated game $G(T)$ in which, for any $t < T$, the outcome of stage t is not a Nash equilibrium of G.

- Consider the following game:

		Player 2		
		L_2	M_2	R_2
Player 1	L_1	1, 1	5, 0	0, 0
	M_1	0, 5	4, 4	0, 0
	R_1	0, 0	0, 0	3, 3

- There are two Nash equilibria: (L_1, L_2) and (R_1, R_2).
- Suppose the game is repeated twice.
- Then it is possible that the first-stage outcome is neither (L_1, L_2) nor (R_1, R_2) in a subgame-perfect Nash equilibrium.
- Consider, for example, player i's strategy:
  - play M_i in the first stage;
  - play R_i if the first-stage outcome is (M_1, M_2); otherwise, play L_i.
- It can be verified that the strategy profile constitutes a subgame-perfect Nash equilibrium, in which the first-stage outcome is (M_1, M_2).

stage2 (stage1 不是 subgame)

→ Verify: {是否是子博弈均衡
是否是总体的均衡.

stage 1: (M_1,M_2)
stage 2: (R_1,R_2) ⇒ (7,7).

Player1 在 stage2 显然不会偏离.

在 stage 1 中. {若选 L_1 ⇒ $(L_1,M_2)+(L_1,L_2)=(6,1)$. X.
若选 R_1 , 显然 worse off.

TIPS. 这个问题里不用考虑上面的策略.

★. Q: 在2阶段博弈中, 有多少个NE?

考

①. 要使第1阶段的结果是(M_1,M_2), 第二阶段的结果应为 ____ ?

- player 2 给定选 M_2. player1 可偏离为 {L_1 ⇒ 可行 ⇒ 目的: 通过给定stage2的策略使偏离不可行.
R_1 ⇒ 不可行 (stage1 -4. stage2 最多+2).
↳ 可能 +2/0/-2. 来自 从(L_1,L_2)偏离到(R_1,R_2)或反之.

第二阶段可能的均衡结果: → 目的: 确保 stage1 不偏离.

意义: 若stage1选了表中对应位置, 则 stage2 play 表中的策略.

5+1<4+3.

✓	(L_1,L_2)	✓
(L_1,L_2)	(R_1,R_2)	✓
✓	✓	✓

→ 由NE. 只可能是(L_1,L_2)或(R_1,R_2).

✓: 任意结果均可确保 stage1 无偏离.
↳ 共2^6个策略组合. 若1选L_1. 则2必须选L_2. 对R_1同理
↳ 例: stage 1: (M_1, M_2)
stage 2 {player1: $(L_1, L_1, R_1, L_1, R_1, L_1, R_1, L_1, R_1)$
player 2: $(L_2, L_2, R_2, L_2, R_2, L_2, R_2, L_2, R_2)$
↳ 固定.
其它6组可变. 但要同时变. (不能有(L_1,R_2)这种结果)

②. 对 stage 1 的不同可能做以上讨论.

核心: 先找到 stage 2 的NE. 计算可能在 stage 2 的"获利".
再在 stage1 里找可能的偏离.
↳ 若无论 stage2 怎么变, 都可通过偏离来变得严格优/劣, 则舍. (e.g. (R_1,L_2)).
最后对偏离做限制 ↳ 通过固定stage 2 的部分策略.

- Infinitely Repeated Games

贴现因子

Def. Let π_t be the payoff in stage t. Given the discount factor $\delta \in (0,1)$, the present value of the infinite sequence of payoffs $\pi_1, \pi_2, \ldots$ is

$$\pi_1 + \delta\pi_2 + \delta^2\pi_3 + \cdots = \sum_{t=1}^{\infty} \delta^{t-1}\pi_t.$$

Def. Given a stage game G, let $G(\infty, \delta)$ denote the **infinitely repeated game** in which G is played forever and players share the discount factor δ. For each t, the outcomes of the $t-1$ preceding plays are observed before the t^{th} stage begins. Each player's payoff in $G(\infty, \delta)$ is the present value of the player's payoffs from the infinite sequence of stage games.

e.g.

- Consider the following infinitely repeated game of Prisoners' Dilemma:
  - In the first stage, the two players play the stage game G and receive payoffs $\pi_{1,1}$ and $\pi_{2,1}$;
  - In stage t, the players observe the actions chosen in the preceding $t-1$ stages, and then play G to receive $\pi_{1,t}$ and $\pi_{2,t}$;
  - The payoff of the infinitely repeated game is the present value of the sequence of payoffs: $\sum_{t=1}^{\infty} \delta^{t-1}\pi_{i,t}$ for player $i = 1, 2$.

		Player 2	
		L_2	R_2
Player 1	L_1	1, 1	5, 0
	R_1	0, 5	4, 4

- There are infinitely many strategies for the players.
- Some common strategies:
  - ✓ ① noncooperative strategy:
    - play L_i in every stage
  - 冷酷的
  - ✓ ② (grim) trigger strategy: 触发策略
    - play R_i in the first stage;
    - in stage t, if the outcome of all $t-1$ preceding stages has been (R_1, R_2), then play R_i; otherwise, play L_i
  - 一报还一报
  - 暂不讨论 {③ tit-for-tat (or tit for two tats) strategy t-1 期对手选什么, t期我选什么
  - ④ carrot-and-stick strategy (or two-phase strategy)

软硬兼施

- Strategies in Infinitely Repeated Games.

- We focus on the first two strategies.
- If both players adopt the noncooperative strategy, then (L_1, L_2) is repeated forever.
- Using a trigger strategy, player i cooperates until someone fails to cooperate, which triggers a switch to noncooperation forever.
- If both players adopt the trigger strategy, then the outcome of the infinitely repeated game is (R_1, R_2) in every stage.
- Question: Is it a Nash equilibrium in the infinitely repeated game where both players adopt the trigger strategy (i.e., cooperation is achieved)?

Claim:

Both players adopting the noncooperative strategy is a Nash equilibrium.

Proof.

- Assume player i plays L_i in every stage.
- Then player j's best response is also "to play L_j in every stage".

Claim:

Both players adopting the trigger strategy is a Nash equilibrium if and only if $\delta \geq 1/4$.

Proof.

- Assume player i has adopted the trigger strategy. We seek to show player j's best response is also to adopt the trigger strategy.
- Case 1: The outcome in a previous stage is not (R_1, R_2). Since player i plays L_i forever, player j's best response is also to play L_j forever.
- Case 2: In the first stage or in a stage where all the preceding outcomes have been (R_1, R_2), if player j plays the trigger strategy, then he should play R_j in this stage, and the outcome from this stage onwards will be (R_1, R_2) in every stage. Thus player j's payoff from this stage onwards is

$$\sum_{t=1}^{\infty} 4 \times \delta^{t-1} = \frac{4}{1-\delta}.$$

- If player j plays L_j in this stage, player i still plays R_i in this stage but L_i forever from the next stage. Thus player j will also play L_j from the next stage onwards. This means player j's payoff from this stage onwards is

$$5 + \sum_{t=1}^{\infty} \delta^t = 5 + \frac{\delta}{1-\delta}.$$

- Therefore, playing the trigger strategy in this case is optimal iff

$$\frac{4}{1-\delta} \geq 5 + \frac{\delta}{1-\delta} \Leftrightarrow \delta \geq 1/4.$$

- Summarizing Cases 1 and 2, the trigger strategies constitute a Nash equilibrium for the game iff $\delta \geq 1/4$.

Claim:

The trigger-strategy Nash equilibrium in the infinitely repeated Prisoners' Dilemma game is subgame perfect.

Proof.

- In an infinitely repeated game, a subgame is characterized by its previous history. The subgames can be grouped as follows:
- (i) Subgames whose previous histories are always a finite sequence of (R_1, R_2).
- (ii) Subgames whose previous histories contain other outcomes different from (R_1, R_2).
- For a subgame in Case (i), the players' strategies in such a subgame are again the trigger strategies, which is a Nash equilibrium for the whole game and thus for the subgame as well.
- For a subgame in Case (ii), the players' strategies are simply to repeat (L_1, L_2) all the time in the subgame, which is also a Nash equilibrium.

- In the Prisoners' Dilemma example, the cooperative outcome, which cannot be achieved in the stage game or in any finitely repeated game, can be sustained if the stage game is played forever.
- The condition is that the discount factor is sufficiently large (or players are sufficiently patient).
- Folk theorem: cooperative equilibria which do not exist in static games can be achieved in repeated games.

intuition: 背叛可以暂时获利. 并且如果对未来很不care, 背叛也可以长期获利.

(偏离带来一期的收益和无穷期损失)

		Player 2	
		L_2	R_2
Player 1	L_1	1, 1	5, 0
	R_1	0, 5	4, 4

- 一次性偏离原理. ⇒ 只改某1个 stage. 其它stage都仍按原策略进行.

  - One-deviation principle: A strategy profile is a subgame-perfect Nash equilibrium if and only if, for each player i and for each subgame, no single deviation would raise player i's payoff in the subgame.

e.g. Prisoners' Dilemma.

		Player 2	
		L_2	R_2
Player 1	L_1	1, 1	5, 0
	R_1	0, 5	4, 4

偏离 冷酷触发

	t	t+1	t+2	……
Player 1	L_1	L_1	L_1	~
Player 2	R_2	L_2	L_2	~

收益: $5+\frac{\delta}{1+\delta}$.

- Feasible Payoff

Def. The payoffs $(x_1, \ldots, x_n)$ are **feasible** in the stage game G if they are a convex combination (i.e., a weighted average, where the weights are all nonnegative and sum to one) of the pure-strategy payoffs of G.

  - In the Prisoners' Dilemma example, all pure-strategy payoffs $(1,1)$, $(0,5)$, $(4,4)$ and $(5,0)$ are feasible.
  - The payoffs $(2.5, 2.5)$ are also feasible, which are a convex combination of the pure-strategy payoffs of $(1,1)$ and $(4,4)$.
  - All feasible payoffs are depicted in the shaded region of Figure 1.

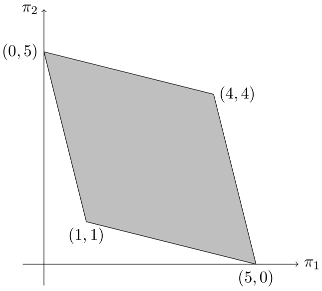


Figure 1: Feasible payoffs in Prisoners' Dilemma

- Average Payoff.

Def. Given the discount factor δ, the **average payoff** of the infinite sequence of payoffs π_1, π_2, … is

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\pi_t.$$

$=\pi=$ 每期固定收益.

t	t+1	~	Sum
π_1	π_2	~	$\sum_{t=1}^{\infty}\delta^{t-1}\pi_t$
π	π	~	$\frac{\pi}{1-\delta}$

$$\Rightarrow \pi=(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\pi_t.$$

- Friedman Theorem.

Thm Let G be a finite, static game of complete information. Let $(e_1, \ldots, e_n)$ denote the payoffs from a Nash equilibrium of G, and let $(x_1, \ldots, x_n)$ denote any feasible payoffs from G, where $x_i > e_i$ for each player i. If the discount factor δ is sufficiently close to one, then there exists a subgame-perfect Nash equilibrium in the infinitely repeated game $G(\infty, \delta)$ that achieves $(x_1, \ldots, x_n)$ as the average payoff.

Intuition: δ 足够大时, 可以得到比NE更好的结果.

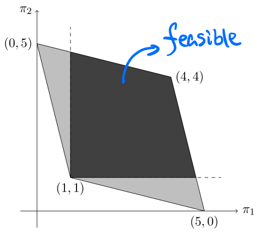


Figure 2: Subgame-perfect Nash equilibria in infinitely repeated games

- 应用: Collusion between Cournot Duopolists

勾结

  - In the Cournot model, the unique Nash equilibrium involves each firm producing $q_c = \frac{a-c}{3}$ and earning a profit of $\pi_c = \frac{(a-c)^2}{9}$.
  - If there is a monopolist, then the monopoly quantity is $q_m = \frac{a-c}{2}$ and the monopoly profit is $\pi_m = \frac{(a-c)^2}{4}$.
  - If the two firms can collude to produce $\frac{q_m}{2}$ each, then they jointly produce the monopoly quantity q_m. Each of them obtains a profit of $\frac{\pi_m}{2} = \frac{(a-c)^2}{8}$.
  - If firm i produces $\frac{q_m}{2}$, then the best response for firm j is to produce $q_d = \frac{3(a-c)}{8}$. In this case, firm i's profit is $\frac{3(a-c)^2}{32}$, while firm j's profit is $\pi_d = \frac{9(a-c)^2}{64}$.
  - Consider the infinitely repeated game based on the Cournot stage game when both firms have the discount factor $0 < \delta < 1$.
  - Trigger strategy:
    - produce half of the monopoly quantity $\frac{q_m}{2}$, in the first period.
    - in period t, produce $\frac{q_m}{2}$ if both firms have produced $\frac{q_m}{2}$ in all the preceding $t-1$ periods; otherwise, produce the Cournot quantity q_c.
  - Here the cooperative output is $\frac{q_m}{2}$ and noncooperative output is q_c.
  - Question: Is the collusive outcome sustained?

Claim: For the infinitely repeated game with the Cournot stage game, both firms playing the trigger strategy is a subgame-perfect Nash equilibrium if and only if $\delta \geq \frac{9}{17}$.

Proof.

- Suppose firm i has adopted the trigger strategy, we need to show firm j's best response is also to player the trigger strategy in any subgame.
- There are again two types of subgames to be checked.

- First, if a quantity other than $\frac{q_m}{2}$ has been chosen by any firm before the current period, then firm i chooses q_c from this period onwards. The best response for firm j is also to choose q_c from this period onwards. Thus, playing the trigger strategy is optimal in this subgame.
- Second, in period t, if the outcomes of all previous periods are $(\frac{q_m}{2}, \frac{q_m}{2})$, firm j's present value of the payoffs from this period onwards if it chooses the trigger strategy is

$$\frac{\pi_m}{2(1-\delta)}.$$

- If firm j deviates from the trigger strategy by choosing a quantity other than $\frac{q_m}{2}$, then firm i produces $\frac{q_m}{2}$ in this period, but q_c from period $t+1$ onwards. Thus, it is optimal for firm j to produce q_d in this period and q_c from period $t+1$ onwards. Thus, firm j's present value of the payoffs from period t onwards is

$$\pi_d + \frac{\delta}{1-\delta}\pi_c.$$

- Therefore, trigger strategy is the best response for firm j to firm i's trigger strategy iff

$$\frac{\pi_m}{2(1-\delta)} \geq \pi_d + \frac{\delta}{1-\delta}\pi_c \Leftrightarrow \delta \geq \frac{\pi_d - \frac{\pi_m}{2}}{\pi_d - \pi_c} = \frac{9}{17}.$$

- What happens if players are less patient, i.e., $\delta < \frac{9}{17}$? Are there any other strategies that can support the collusive outcome as a subgame-perfect Nash equilibrium? ⇒ Yes. 但策略要改变.
- Consider the two-phase (or carrot-and-stick) strategy:
  - in the first period, produce half of the monopoly quantity $\frac{q_m}{2}$;
  - in period t, produce $\frac{q_m}{2}$ if both firms produce $\frac{q_m}{2}$ or both firms produce x in period $t-1$; otherwise, produce x.
- This strategy involve a (one-period) punishment phase in which the firm produces x and a (potentially infinite) collusive phase in which the firm produces $\frac{q_m}{2}$.
- Such a strategy punishes
  - a firm for deviating from the collusive phase
  - a firm for deviating from the punishment phase (不惩罚也 worse off)
- If both firms produce x, the profit of each firm is denoted by $\pi(x) = (a-2x-c)x$, where $\frac{x}{a-c} \leq \frac{1}{2}$.
- If firm i produces x, the best response of firm j is to produce $q_{dp} = \frac{a-x-c}{2}$ and the corresponding profit is denoted by $\pi_{dp}(x) = \frac{(a-x-c)^2}{4}$.
- There are two types of subgames:
  - (i) collusive subgames: the outcome of previous period is either $(\frac{q_m}{2}, \frac{q_m}{2})$ or (x, x);
  - (ii) punishment subgames: the outcome of previous period is neither $(\frac{q_m}{2}, \frac{q_m}{2})$ nor (x, x).
- To show both firms adopting the two-phase strategy is a subgame-perfect Nash equilibrium, we use the one-deviation principle.
- Suppose firm i has adopted the two-phase strategy.
- In collusive subgames, if firm j also adopts the two-phase strategy, its payoff is 从($\frac{q_m}{2}$, $\frac{q_m}{2}$)开始

$$\left(1+\delta+\frac{\delta^2}{1-\delta}\right)\frac{1}{2}\pi_m. = \frac{1}{1-\delta}\cdot\frac{1}{2}\pi_m.$$

从($\frac{q_m}{2}$, q_d)开始

- If firm j deviates in this period only, then firm i still chooses $\frac{q_m}{2}$ in this period but x in the next period. Then firm j would choose q_d in this period and x in the next period. The payoff from deviation is

$$\underset{\text{不合作}}{\pi_d} + \underset{\text{punish}}{\delta\pi(x)} + \underset{\text{后面都合作}}{\frac{\delta^2}{1-\delta}\frac{1}{2}\pi_m}.$$

[接右]

[续左]

- Thus, choosing the two-phase strategy is optimal iff

$$(1+\delta)\frac{1}{2}\pi_m \geq \pi_d + \delta\pi(x). \qquad (1)$$

对违反者的要求

从(x,x)开始

- In punishment subgames, it is optimal to choose the two-phase strategy for firm j iff

第一期不惩罚而选择最优化

$$\pi(x) + \delta\frac{1}{2}\pi_m \geq \underline{\pi_{dp}(x)} + \delta\pi(x). \qquad (2)$$

对遵守者的要求

- Both firms adopting the two-phase strategy is a subgame-perfect Nash equilibrium iff (1) and (2) hold.
- The two conditions (1) and (2) can be rewritten as

$$\delta\left(\frac{1}{2}\pi_m - \pi(x)\right) \geq \pi_d - \frac{1}{2}\pi_m, \qquad (3)$$

$$\delta\left(\frac{1}{2}\pi_m - \pi(x)\right) \geq \pi_{dp}(x) - \pi(x). \qquad (4)$$

- Intuitions: the gain this period from deviating must not exceed the discounted value of the loss next period from punishment.
- Consider the case $\delta = \frac{1}{2} < \frac{9}{17}$.
- Condition (3) is satisfied iff $\frac{x}{a-c} \leq \frac{1}{8}$ or $\frac{x}{a-c} \geq \frac{3}{8}$.
- Condition (4) is satisfied iff $\frac{3}{10} \leq \frac{x}{a-c} \leq \frac{1}{2}$.
- Thus, two-phase strategies constitute a subgame-perfect Nash equilibrium in the game iff $\frac{3}{8}(a-c) \leq x \leq \frac{1}{2}(a-c)$.

⇒ $\delta < \frac{9}{17}$ 时子博弈精炼解是否存在依赖于x的取值.

Lec 9. Bargaining Games. (we focus on 2 players situation)

最后通牒

• Ultimatum Games

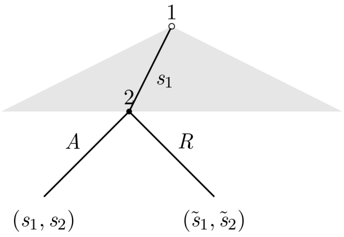


Figure 1: A game tree for the ultimatum game

- Suppose player 1 makes an offer (s_1, s_2), where s_i is the share for player $i = 1, 2$, and $s_1 + s_2 = 1$.
- After observing the offer from player 1, player 2 decides whether to accept the offer or not.
- If the offer is accepted, then each player i receives s_i; otherwise, there is an exogenous settlement (支付) $(\tilde{s}_1, \tilde{s}_2)$ which involves player i receiving $\tilde{s}_i$, where $\tilde{s}_1 + \tilde{s}_2 < 1$.
- For instance, $\tilde{s}_1 = \tilde{s}_2 = 0$ means that both players receive nothing if no agreement is reached.

- Player 2 has infinite strategies. e.g. $\begin{cases} A, & \text{if } s_2 \geq \frac{1}{2} \\ R, & \text{if } s_2 < \frac{1}{2} \end{cases}$.

  ↗ 来自 $\tilde{s}_1+\tilde{s}_2$ 与1 的差. 在这个区间里都是NE.

- There are infinitely many NE.

- But a unique subgame-perfect NE: player 1 makes $(1-\tilde{s}_2, \tilde{s}_2)$

  player 2 $\begin{cases} A, & \text{if } s_2 \geq \tilde{s}_2 \\ R, & \text{if } s_2 < \tilde{s}_2 \end{cases}$ (若 $s_2 > \tilde{s}_2$, 则找不到 player 1 的 best response, NE不存在)

交替出价

• Alternating - Offer Game → player 2 也可以出价了

☆. TIPS. 不是重复博弈!

↗ 原因: 每期的 game 不一样.

- Suppose now the two players make alternating offers in each period.
- The common discount factor is $0 < \delta < 1$. 有贴现
- The three-period bargaining game is:
  - (1a) In the first period, player 1 proposes $s_1(1)$ for himself and $s_2(1)$ for player 2.
  - (1b) Player 2 either accepts the offer to end the game or rejects the offer to continue the game.
  - (2a) In the second period, player 2 proposes $s_1(2)$ for player 1 and $s_2(2)$ for himself.
  - (2b) Player 1 either accepts the offer to end the game or rejects the offer to continue the game.
  - (3) In the third period, player 1 receives a share of s and player 2 receives $1 - s$, where $0 < s < 1$.

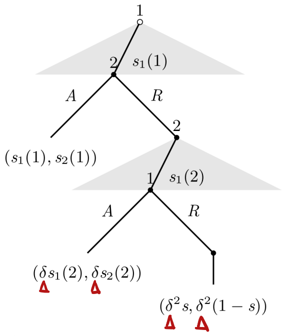


Figure 2: A game tree for the alternating-offer game

- Let $s_1(3) = s$ and $s_2(3) = 1 - s$.
- In general, in period t, $s_1(t)$ and $s_2(t)$ are offered to players 1 and 2, where the offers satisfy

$$s_1(t) + s_2(t) = 1.$$

- The present value of payoff to player i is $\delta^{t-1} s_i(t)$ if the bargaining game is ended in period t.

- We use backwards induction to solve the game.
- In the second period, player 2 is at the move. Because the payoff to player 1 in period 3 is s, player 2 will offer $s_1(2) = \delta s$ to player 1 and $s_2(2) = 1 - \delta s$ to himself. Player 1 accepts the offer.

  ↗ 每个阶段都是1个 ultimatum 形式的 subgame.

- In the first period, player 1 will offer $\delta(1 - \delta s)$ to player 2 and $1 - \delta(1 - \delta s)$ to himself. Player 2 will accept the offer and the game ends.
- The unique backwards-induction outcome of the three-period game is:
  - Player 1 offers the settlement

$$\begin{aligned} s_1^*(1) &= 1 - \delta(1 - \delta s), \\ s_2^*(1) &= \delta(1 - \delta s). \end{aligned}$$

  - Player 2 accepts the offer.
  - The game ends in period 1.

• Alternating-Offer Game with Infinite Periods.

↳ 逆向归纳法无效.

- Suppose the alternating-offer game is repeated forever until one player accepts an offer.
- The infinite-period game is the same as the three-period bargaining game except that the exogenous settlement in step (3) is replaced by an infinite sequence of steps (3a), (3b), (4a), (4b) and so on.
- The game beginning in period 3 is identical to the game beginning in period 1, because it is also an infinite-period game.

→ 原理：第3期和第1期是相同的博弈.

↗仅无限期时可这么做.

- If there is a unique backwards-induction outcome s, then $f(s) = s$, where $f(s) = 1 - \delta(1 - \delta s)$ and hence $s = \frac{1}{1+\delta}$.
- We aim to show the backwards-induction outcome is indeed unique.
- Let s_h be the highest payoff player 1 can receive in any backwards-induction outcome of the game as a whole.
- We can also regard s_h as the third-period payoff for player 1. Then the result of the three-period bargaining game says that, using s_h as the exogenous settlement s,

$$f(s_h) = 1 - \delta(1 - \delta s_h)$$

  is a payoff for player 1 in period 1.

$f(s_h)$：第1期 player 1 的收入

s_h：-3 - ——— ——

- Hence $f(s_h) \le s_h$, since s_h is also the maximum payoff in period 1.
- Because any first-period payoff for player 1 can be represented in the form of $f(s)$ with some third-period payoff s, there exists a s_3 such that $s_h = f(s_3)$. Because $f(s)$ is an increasing function of s and $s_3 \le s_h$, then $s_h \le f(s_h)$. Therefore, we must have $s_h = f(s_h)$.
- Let s_l be the lowest payoff player 1 can receive in any backwards-induction outcome of the game as a whole. Similarly, $f(s_l) = s_l$.
- Solving $f(s) = s$, we obtain a unique solution $s = \frac{1}{1+\delta}$. ↗只有一个解 ⟺ max = min.
- Therefore $s_h = s_l = \frac{1}{1+\delta}$, which implies that $s^* = \frac{1}{1+\delta}$ is the unique backwards-induction outcome.
- The unique backwards-induction outcome is:
  - In the first period, player 1 offers $s^* = \frac{1}{1+\delta}$ to himself and $1 - s^*$ to player 2.
  - Player 2 accepts the offer.
  - The game ends.
- The game has infinitely many periods, but ends at the first period.
- The player with the first move gains a higher payoff (i.e., first-mover advantage).

Lec 10. Static Games of Incomplete Information

· Introduction

- In the auction example, each player's payoff function is no longer common knowledge ⇒ buyer i's payoff function is not known by other buyers.
- This is an example of **incomplete information** games, in which at least one player is uncertain about another player's payoff function.
- Games of incomplete information are also called **Bayesian games**.
- Two types of Bayesian games: static vs. dynamic

· Cournot Competition under Asymmetric Information

- Consider the Cournot duopoly model with an inverse demand function $P = a - Q$, where $Q = q_1 + q_2$ and $a > 0$.
- Firm 1's cost function is $c_1(q_1) = cq_1$.
- Firm 2's cost function is

$$c_2(q_2) = \begin{cases} c_H q_2, & \text{with probability } \theta, \\ c_L q_2, & \text{with probability } 1-\theta, \end{cases}$$

  where $c_L < c_H$ and $0 < \theta < 1$.
- Different from the standard Cournot model, the information is asymmetric:
  - Firm 1's cost function is known by both firms ⇒ $c_1(\cdot)$ is common knowledge.
  - Firm 2's cost function is completely known by itself, but not known by firm 1 ⇒ $c_2(\cdot)$ is not common knowledge. → 2知道自己是 C_H or C_L
  - Firm 1 only knows the distribution of firm 2's marginal cost, i.e., c_H with probability θ and c_L with probability $1-\theta$. → 1知道 2 C_H和C_L的概率.
- What will be the quantities chosen by the firms?
- Naturally, firm 2 may want to choose a different (and presumably lower) quantity if its marginal cost is high than if it is low.
- Firm 1 should rationally anticipate that firm 2 may tailer its quantity to its cost in this way.
- Let $q_2^*(c_H)$ and $q_2^*(c_L)$ denote firm 2's quantity choices when its marginal cost is c_H and c_L respectively, and let q_1^* denote firm 1's single choice of quantity.
- If firm 2's cost is c_j $(j = L, H)$, it will choose $q_2^*(c_j)$ to solve

$$\max_{q_2}(a - q_1^* - q_2 - c_j)q_2.$$

- Since firm 1 knows that firm 2's marginal cost is c_H with probability of θ and anticipates firm 2 to choose $q_2^*(c_j)$ depending on its cost, firm 1 chooses q_1^* to solve

$$\max_{q_1} \theta(a - q_1 - q_2^*(c_H) - c)q_1 + (1-\theta)(a - q_1 - q_2^*(c_L) - c)q_1.$$

- The (interior) first-order conditions (or best response functions) for the firms are

$$\begin{aligned} q_2^*(c_H) &= \frac{a - q_1^* - c_H}{2}, \\ q_2^*(c_L) &= \frac{a - q_1^* - c_L}{2}, \\ q_1^* &= \frac{a - \theta q_2^*(c_H) - (1-\theta)q_2^*(c_L) - c}{2}. \end{aligned}$$

- The equilibrium of this game is $(q_1^*, (q_2^*(c_H), q_2^*(c_L)))$, where

$$\begin{aligned} q_1^* &= \frac{a - 2c + \theta c_H + (1-\theta)c_L}{3}, \\ q_2^*(c_H) &= \frac{a - 2c_H + c}{3} + \frac{1-\theta}{6}(c_H - c_L), \\ q_2^*(c_L) &= \frac{a - 2c_L + c}{3} - \frac{\theta}{6}(c_H - c_L). \end{aligned}$$

- We know $q_2^*(c_H) < q_2^*(c_L)$ ⇒ firm 2 produces less when its marginal cost increases.
- Firm 2 has two payoff functions

$$\begin{aligned} \pi_2(q_1, q_2; c_L) &= (a - q_1 - q_2 - c_L)q_2, \\ \pi_2(q_1, q_2; c_H) &= (a - q_1 - q_2 - c_H)q_2. \end{aligned}$$

- Firm 1 has only one payoff function

$$\pi_1(q_1, q_2; c) = (a - q_1 - q_2 - c)q_1.$$

e.g. Auction.

- Suppose a seller wants to sell a product among a group of buyers.
- Each buyer is willing to pay v_i for the product, where v_i is buyer i's private information, i.e., only buyer i knows its valuation v_i, but not all other buyers or the seller.
- In order to sell the product, the seller runs an auction (e.g., first-price, second-price).
- Each buyer must bid for the product in order to be the winner.

- Firm 2 knows firm 1's payoff function, while firm 1 does not know firm 2's payoff functions but only knows the probability distribution.
- This is an example of (static) Bayesian games.

· Static Bayesian Games

- Consider a general static Bayesian game.
- Let player i's possible payoff function be $u_i(a_1, \ldots, a_n; t_i)$, where a_i is player i's action and t_i is called player i's **type**, which belongs to a set of possible types T_i (or **type space**).
- Player i's type t_i is his private information, and each type t_i corresponds to a different payoff function of player i.
- Let $t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ be the types of other players and T_{-i} be the set of all t_{-i}.
- Player i is uncertain about other players' types, but only knows the probability distribution $p_i(t_{-i}|t_i)$ on T_{-i}, which is i's **belief** about other players' types, given i's knowledge of his own t_i.

条件概率!

Def. The **normal-form representation** of an n-player static Bayesian game specifies players'

1) action spaces $A_1, \ldots, A_n$,
2) type spaces $T_1, \ldots, T_n$,
3) beliefs $p_1, \ldots, p_n$,
4) payoff functions $u_1, \ldots, u_n$.

We denote this game by

$$G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n\}.$$

e.g.
- In the Cournot game with asymmetric information,
  - $A_1 = A_2 = [0, \infty)$;
  - $T_1 = \{c\}$, and $T_2 = \{c_H, c_L\}$;
  - $p_1(c_H|c) = \theta, p_1(c_L|c) = 1 - \theta$, and $p_2(c|c_H) = p_2(c|c_L) = 1$;
  - Payoff functions are

$$\begin{aligned} \pi_1(q_1, q_2; c) &= (a - q_1 - q_2 - c)q_1, \\ \pi_2(q_1, q_2; c_L) &= (a - q_1 - q_2 - c_L)q_2, \\ \pi_2(q_1, q_2; c_H) &= (a - q_1 - q_2 - c_H)q_2. \end{aligned}$$

时序

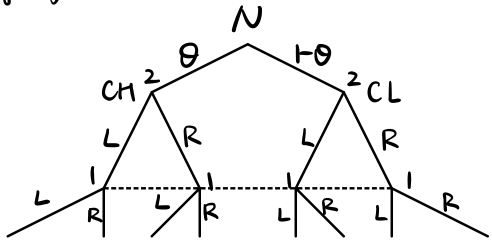
- The timing of a static Bayesian game:
  1. Nature draws a type vector $t = (t_1, \ldots, t_n)$, where $t_i \in T_i$;
  2. Nature reveals t_i to player i, but not to any other players;
  3. The players simultaneously choose actions, player i choosing $a_i \in A_i$;
  4. Payoffs $u_i(a_1, \ldots, a_n; t_i)$ are received.

类似狼人杀的上帝.

- By introducing the frictional moves by nature in (1) and (2), we have described a game of incomplete information as a game of imperfect information.
- We often assume that the nature draws $t = (t_1, \ldots, t_n)$ according to the prior probability distribution $p(t)$, which is common knowledge.
- Then the belief $p_i(t_{-i}|t_i)$ can be computed by **Bayes' rule**

$$p_i(t_{-i}|t_i) = \frac{p(t_{-i}, t_i)}{\underbrace{\sum_{t'_{-i} \in T_{-i}} p(t'_{-i}, t_i)}_{= p(t_i)}}.$$

e.g. for Cournot Game above.



- Two remarks:
- First, there are games in which player i has private information not only about his or her own payoff function but also about another player's payoff function. We write player i's payoff function as $u_i(a_1, \ldots, a_n; t_1, \ldots, t_n)$.

→ 某个player可以掌握其它人的信息.

- Second, we typically assume that players' types are independent, i.e., $p_i(t_{-i}|t_i)$ does not depend on t_i, which can be denoted by $p_i(t_{-i})$. But $p_i(t_{-i})$ is still derived from the prior distribution $p(t)$.

Def. In the static Bayesian game $G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n\}$, a **strategy** for player i is a function $s_i(t_i)$, i.e., $s_i : T_i \to A_i$. For given type t_i, $s_i(t_i)$ gives an action of player i. Player i's **strategy space** S_i is the set of all functions from T_i into A_i.

不同类型的player会怎么行动.

- In the previous example, $(q_2^*(c_H), q_2^*(c_L))$ is a strategy for firm 2, while q_1^* is a strategy for firm 1.

• Bayesian NE

Def. In the static Bayesian game $G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n\}$, the strategies $s^* = (s_1^*, \ldots, s_n^*)$ are a (pure-strategy) Bayesian Nash equilibrium if for each player i and for each of i's types $t_i \in T_i$, $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} \sum_{\underline{t_{-i} \in T_{-i}}} u_i(s_{-i}^*(t_{-i}), a_i; t_i) \underline{p_i(t_{-i}|t_i)}.$$

对ti以外的类型 求期望.

- In a general finite static Bayesian game (finite players, finite actions, and finite types), a Bayesian Nash equilibrium exists, perhaps in mixed strategies.

- In a Bayesian Nash equilibrium, each player's strategy is a best response to other players' strategies.
- In other words, no player wants to change his or her strategy unilaterally given other players' equilibrium strategies, even if the change involves only one action by one type.

  单方面地

- A Bayesian Nash equilibrium is simply a Nash equilibrium in a Bayesian game.
- In the Cournot game with asymmetric information, the strategies $(q_1^*, (q_2^*(c_H), q_2^*(c_L)))$ are a Bayesian Nash equilibrium since neither firm 1 nor firm 2 wants to deviate from its equilibrium strategy.

• Mixed Strategies Revisited.

- Consider the game of battle of the sexes

		Wife	
		Opera	Football
Husband	Opera	1, 2	0, 0
	Football	0, 0	2, 1

- There are three possible Nash equilibria: (Opera, Opera), (Football, Football) and ($\frac{1}{3}$Opera+$\frac{2}{3}$Football, $\frac{2}{3}$Opera+$\frac{1}{3}$Football).
- In the mixed-strategy Nash equilibrium, the husband plays Opera with probability 1/3 and Football with probability 2/3, while the wife plays Opera with probability 2/3 and Football with probability 1/3.
- Suppose the couple are uncertain about the payoffs for each other.
- Consider the following payoff matrix

		Wife	
		Opera	Football
Husband	Opera	$1, 2 + t_w$	$0, 0$
	Football	$0, 0$	$2 + t_h, 1$

- Here t_w is privately known by the wife, while t_h is privately known by the husband.
- Assume that t_w and t_h are independently drawn from a uniform distribution on $[0, x]$, where $x > 0$.

  选出

- The normal-form representation of this static Bayesian game is $G = \{A_h, A_w; T_h, T_w; p_h, p_w; u_h, u_w\}$:
  - $A_h = A_w = \{\text{Opera, Football}\}$;
  - $T_h = T_w = [0, x]$;
  - The husband believes that t_w (the wife believes that t_h) is uniformly distributed on $[0, x]$;
  - u_h and u_w are given before.
- What are players' strategies?
- We can construct a Bayesian Nash equilibrium (s_h^*, s_w^*), where

$$s_h^* = \begin{cases} \text{Football}, & \text{if } t_h > \bar{t}_h, \\ \text{Opera}, & \text{if } t_h \le \bar{t}_h, \end{cases} \text{ and } s_w^* = \begin{cases} \text{Opera}, & \text{if } t_w > \bar{t}_w, \\ \text{Football}, & \text{if } t_w \le \bar{t}_w. \end{cases}$$

- Note $\bar{t}_h$ and $\bar{t}_w$ are two critical values, which need to be determined.
- In the Bayesian Nash equilibrium, the husband will choose Football if t_h exceeds the critical value $\bar{t}_h$, and choose Opera otherwise.
- Given the wife's strategy, the husband's expected payoffs of choosing Opera and Football are

$P(t_w > \bar{t}_w) = 1 - F(\underline{\bar{t}_w})$ 临界值. 目前未知.

$$\begin{aligned} u_h(\text{Opera}, s_w^*|t_h) &= \Pr(s_w^* = \text{Opera}) \cdot 1 + \Pr\,(s_w^* = \text{Football}) \cdot 0 \\ &= \left(1 - \frac{\bar{t}_w}{x}\right) \cdot 1 + \frac{\bar{t}_w}{x} \cdot 0 = 1 - \frac{\bar{t}_w}{x}, \end{aligned}$$

and

$$u_h(\text{Football}, s_w^*|t_h) = \left(1 - \frac{\bar{t}_w}{x}\right) \cdot 0 + \frac{\bar{t}_w}{x} \cdot (2 + t_h) = \frac{\bar{t}_w}{x}(2 + t_h).$$

- Thus, choosing Opera is optimal iff

$$1-\frac{\bar{t}_w}{x}\geq\frac{\bar{t}_w}{x}(2+t_h)\Leftrightarrow t_h\leq\bar{t}_h=\frac{x}{\bar{t}_w}-3. \qquad (1)$$

- Similarly, given the husband's strategy, the wife's expected payoffs of playing Opera and Football are

$$u_w(\text{Opera},s_h^*|t_w)=\frac{\bar{t}_h}{x}\cdot(2+t_w)+\left(1-\frac{\bar{t}_h}{x}\right)\cdot 0=\frac{\bar{t}_h}{x}(2+t_w),$$

and

$$u_w(\text{Football},s_h^*|t_w)=\frac{\bar{t}_h}{x}\cdot 0+\left(1-\frac{\bar{t}_h}{x}\right)\cdot 1=1-\frac{\bar{t}_h}{x}.$$

(这里用Opera也一样)

- Thus, choosing Football is optimal iff

$$1-\frac{\bar{t}_h}{x}\geq\frac{\bar{t}_h}{x}(2+t_w)\Leftrightarrow t_w\leq\bar{t}_w=\frac{x}{\bar{t}_h}-3. \qquad (2)$$

→ husband 最优的条件
+ wife 最优的条件.
⇓
均满足时.即为临界值. 二人均最优.

- Solving (1) and (2) simultaneously, we obtain $\bar{t}_h=\bar{t}_w=\frac{\sqrt{9+4x}-3}{2}$.
- In equilibrium, the husband plays Opera with probability p^* and Football with probability $1-p^*$, while the wife plays Football with probability p^* and Opera with probability $1-p^*$, where

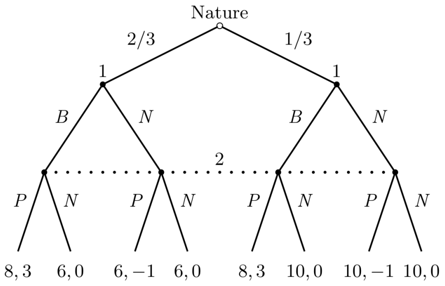
$$p^*=\frac{\bar{t}_h}{x}=\frac{\bar{t}_w}{x}=\frac{2}{\sqrt{9+4x}+3}.$$

- When $x\to 0$, we get that $p^*\to\frac{1}{3}$.
- As the incomplete information disappears, the players' behavior in this pure-strategy Bayesian Nash equilibrium approaches their behavior in the mixed-strategy Nash equilibrium in the original game of complete information.

• A Trading Game.

- Suppose a seller can procure (获得) a product at a cost of $c=1$.
- A buyer wants to buy the good, and is willing to pay $v_0=12$.
- The buyer can also purchase the good from other places, where the valuation is his private information.
- The seller knows that the distribution of the valuation for the outside option is either $v=10$ or $v=14$, each with a probability of 2/3 and 1/3, respectively.
- The price of the good is $p=4$, which is exogenous and independent of where the buyer makes a purchase.
- All c, v_0 and p are common knowledge among both players.
- The seller decides whether to procure the good, and the buyer simultaneously decides whether to order the good from the seller.
- If the seller procures the good, its payoff is $p-c$ if the buyer makes a purchase, and $-c$ otherwise.
- If the seller does not procure the good, its payoff is zero regardless of the buyer's choice.
- The buyer's payoff is v_0-p if he buys from the seller, and $v-p$ otherwise.
- What should the seller and the buyer do?

- The extensive-form representation of the game:



- Player 1 is the buyer and player 2 is the seller.
- Normal-form representation of the game:
  - Action spaces: $A_1=\{B,N\}$ and $A_2=\{P,N\}$;
  - Type spaces: $T_1=\{10,14\}$ and $T_2=\{1\}$;
  - Beliefs: the buyer's belief on the seller's type is 1 on $\{1\}$, and the seller's belief on the buyer's types is 2/3 on 10 and 1/3 on 14;
  - Payoffs are given as above.
- Strategy spaces: $S_1=\{BB,BN,NB,NN\}$ and $S_2=\{P,N\}$
  - The meaning of BN: the buyer with outside option 10 chooses "to buy" and with outside option 14 chooses "not to buy".
- Alternatively, we can use the following matrix to represent the game:

		Buyer			
		BB	BN	NB	NN
Seller	P	$3,8,8$	$5/3,8,10$	$1/3,6,8$	$-1,6,10$
	N	$0,6,10$	$0,6,10$	$0,6,10$	$0,6,10$

这么写的原因: 决策时. Seller 可用的信息只有期望. 而 Buyer 知道自己的类型 并据此做决策.

- For example, consider the outcome (P,BN):
  - the buyer with type 10 receives $v_0-p=8$, and with type 14 receives $v-p=10$;
  - the seller's expected payoff is $3\times 2/3-1\times 1/3=5/3$.
- In particular, we can consider two types of the buyer as two players and we can solve the Bayesian Nash equilibria in the above (like three-player) normal-form representation of the game.
- We first find out the best response functions for each of the "three players" (the seller and each type of the buyer).

		Buyer			
		BB	BN	NB	NN
Seller	P	$\underline{3},\underline{8},8$	$\underline{5/3},\underline{8},\underline{10}$	$\underline{1/3},6,8$	$-1,6,\underline{10}$
	N	$0,\underline{6},\underline{10}$	$0,\underline{6},\underline{10}$	$0,\underline{6},\underline{10}$	$\underline{0},\underline{6},\underline{10}$

- Two Bayesian Nash equilibria: (P,BN) and (N,NN).

Lec 11 Auctions

• Introduction

- One of the most popular examples of static games of incomplete information is an auction.
- An auction is a mechanism of allocating goods.
- Advantages of auctions:
  - a simple way of determining the market-based prices
  - more flexible than setting a fixed price
  - can usually achieve efficiency by allocating the resources to those who value them most highly

• Types of Auctions

- Number of objects
  - A single object or many?
- Open vs. sealed-bid 密封报价
  - Do you know the bids of other participants?
- One-sided vs. two-sided
  - Do buyers and sellers both submit bids, or just buyers?
- Private value vs. common value
  - Do bidders have the same valuation for the object?

• 4 Classical Auctions

- English: ascending (上升的), open
- Dutch: descending (下降的), open
- First-price, sealed-bid
- Second-price, sealed-bid (or Vickrey)

↳ 赢家支付第二高的报价.

• A Second - Price Sealed - Bid Auction.

- Suppose there are n potential buyers (or bidders), with valuations $v_1, \ldots, v_n$ for an object.
- Suppose v_i belongs to the set V_i for all i.
- Bidders know their own valuation but do not know other bidders' valuations.
- The bidders simultaneously submit bids $b_i \in [0, \infty)$.
- The highest bidder wins the object and pays the second highest bid, while the other bidders obtain nothing.
- If there are more than one winners, the object is allocated randomly among them.
- Let r_i be the highest bid of all players other than player i, where $r_i = \max_{j \neq i} b_j$.
- The bidder i's payoff function is

$$u_i(b_i, b_{-i}; v_i) = \begin{cases} v_i - r_i, & \text{if } b_i > r_i; \\ \frac{v_i - r_i}{k}, & \text{if } b_i = r_i; \\ 0, & \text{if } b_i < r_i, \end{cases}$$

where k is the number of bids that equal b_i.

Prop.

Consider a strategy profile $(s_1^*, \ldots, s_n^*)$ in a static Bayesian game. Suppose for any player i, any $t_i \in T_i$, $a_i \in A_i$, and $a_{-i} \in A_{-i}$,

$$u_i(s_i^*(t_i), a_{-i}; t_i) \geq u_i(a_i, a_{-i}; t_i),$$

(i.e., $s_i^*(t_i)$ weakly dominates every $a_i \in A_i$). Then $(s_1^*, \ldots, s_n^*)$ is a Bayesian Nash equilibrium.

→ 弱占优 ⇒ NE. (但不一定唯一)

- **Proof:** Because $s_{-i}^*(t_{-i}) \in A_{-i}$, the weak dominance implies

$$u_i(s_i^*(t_i), s_{-i}^*(t_{-i}); t_i) \geq u_i(a_i, s_{-i}^*(t_{-i}); t_i)$$

for any $t_i \in T_i$ and $a_i \in A_i$.

- Then $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i(a_i, s_{-i}^*(t_{-i}); t_i) p_i(t_{-i} | t_i),$$

for all t_i and for all i.

- Therefore, $(s_1^*, \ldots, s_n^*)$ is a Bayesian Nash equilibrium.

- Each player i's strategy is a function s_i from V_i into $[0, \infty)$.
- For player i, consider the strategy of bidding his true valuation s_i^*, where $s_i^*(v_i) = v_i$ for all $v_i \in V_i$.
- We can show that for any v_i, $s_i^*(v_i) = v_i$ weakly dominates all other bids.
- First, compare $s_i^*(v_i) = v_i$ with $\underset{\downarrow\ \text{“我”的报价}}{b_i} > v_i$:

别人的报价

$$u_i(v_i, b_{-i}; v_i) = \begin{cases} 0, \\ 0, \\ 0, \\ v_i - r_i \end{cases} \geq u_i(b_i, b_{-i}; v_i) = \begin{cases} 0, & \text{if } \underline{r_i} > b_i; \\ \frac{v_i - r_i}{k}, < 0 & \text{if } r_i = b_i; \\ v_i - r_i, < 0 & \text{if } v_i < r_i < b_i; \\ v_i - r_i, & \text{if } r_i \leq v_i. \end{cases}$$

- Then $s_i^*(v_i) = v_i$ weakly dominates $b_i > v_i$.
- Second, compare $s_i^*(v_i) = v_i$ with $b_i < v_i$:

$$u_i(v_i, b_{-i}; v_i) = \begin{cases} 0, \\ v_i - r_i, \\ v_i - r_i, \\ v_i - r_i \end{cases} \geq u_i(b_i, b_{-i}; v_i) = \begin{cases} 0, & \text{if } r_i \geq v_i; \\ 0, & \text{if } b_i < r_i < v_i; \\ \frac{v_i - r_i}{k}, & \text{if } r_i = b_i; \\ v_i - r_i, & \text{if } r_i < b_i. \end{cases}$$

- Then $s_i^*(v_i) = v_i$ weakly dominates $b_i < v_i$.
- Since $s_i^*(v_i) = v_i$ weakly dominates all b_i for any v_i and any player i, by the previous proposition, $(s_1^*, \ldots, s_n^*)$ is a Bayesian Nash equilibrium.

• A First-Price Sealed-Bid Auction

- Suppose there are two bidders: $i = 1, 2$.
- The bidders' valuations for an object are v_1 and v_2, which are independently and uniformly distributed on $[0, 1]$.
- The valuation v_i is bidder i's private information, which is unknown to the other bidder.
- Bidders submit their bids b_1 and b_2 simultaneously.
- The higher bidder wins the object and pays the highest bid, while the other obtains nothing.
- If there is a tie, the winner is determined by a flip of a coin.
- The normal-form representation of this static Bayesian game is $G = \{A_1, A_2; T_1, T_2; p_1, p_2; u_1, u_2\}$:
  - $A_1 = A_2 = [0, \infty)$, and each bid is $b_i \in A_i$;
  - $T_1 = T_2 = [0, 1]$, and each valuation is $v_i \in T_i$;
  - Player i believes that v_j is uniformly distributed on $[0, 1]$;
  - The payoff $u_i(b_i, b_j; v_i)$ is

$$u_i(b_i, b_j; v_i) = \begin{cases} v_i - b_i, & \text{if } b_i > b_j; \\ \frac{1}{2}(v_i - b_i), & \text{if } b_i = b_j; \\ 0, & \text{if } b_i < b_j. \end{cases}$$

- Bidder i's strategy is a function $s_i(v_i)$ from $[0, 1]$ into $[0, \infty)$.
- (s_1^*, s_2^*) is a Bayesian Nash equilibrium if and only if for $i = 1, 2$ and each $v_i \in [0, 1]$, $s_i^*(v_i)$ solves

$$\begin{aligned} & \max_{b_i \geq 0} E_{v_j} u_i(b_i, s_j^*(v_j); v_i) \\ = & \max_{b_i \geq 0} \left\{ (v_i - b_i)\Pr\{b_i > s_j^*(v_j)\} + \frac{1}{2}(v_i - b_i)\overset{0\ (\text{连续型 r.v. 的情形})}{\overset{\shortparallel}{\Pr\{b_i = s_j^*(v_j)\}}} \right\} \end{aligned}$$

- There may be many Bayesian Nash equilibria in this game.
- We focus on equilibria in the form of linear functions:

$$s_1^*(v_1) = a_1 + c_1 v_1, \text{ and } s_2^*(v_2) = a_2 + c_2 v_2,$$

  where $c_i > 0$, and $0 \leq a_i < 1$ for $i = 1, 2$.
- To solve for the Bayesian Nash equilibria, we just need to find out the coefficients a_i and c_i accordingly.
- Rationale of the assumptions on a_i and c_i:
  - $c_i > 0$: a bidder with higher valuation is willing to bid higher
  - $a_i \geq 0$: bids cannot be negative
  - $a_i < 1$: for $a_i \geq 1$, bidder i can never end up with a positive payoff given $v_i \in [0, 1]$
- We need to determine each player's best response given the other's strategy.
- Suppose player j adopts a linear strategy $s_j^*(v_j) = a_j + c_j v_j$ in equilibrium, where $c_j > 0$.
- We have

$$\Pr(b_i = a_j + c_j v_j) = \Pr\left(v_j = \frac{b_i - a_j}{c_j}\right) = 0.$$

连续的情形下.
单点概率为0.

- For any $v_i \in [0, 1]$, player i's best response b_i maximizes

$$(v_i - b_i)\Pr(b_i > a_j + c_j v_j) = (v_i - b_i)\underline{\Pr\left(v_j < \frac{b_i - a_j}{c_j}\right)}. = \frac{b_i - a_j}{c_j}$$

法2.(更简单)

- Alternatively, if we can somehow guess that $(s_1^*(v_1), s_2^*(v_2)) = (v_1/2, v_2/2)$ is a Bayesian Nash equilibrium, we can prove it directly.
- Suppose player j has adopted the strategy $s_j^*(v_j) = v_j/2$.
- Player i's best response b_i solves $= \Pr(v_j < 2b_i) = \frac{2b_i}{1-0}$

$$\max_{b_i \in [0,1/2]} (v_i - b_i)\Pr(b_i > v_j/2) = \max_{b_i \in [0,1/2]} 2(v_i - b_i)b_i.$$

- For any $v_i \in [0, 1]$, the unique maximizer is $b_i^* = v_i/2$.
- Thus $(s_1^*(v_1), s_2^*(v_2)) = (v_1/2, v_2/2)$ is a Bayesian Nash equilibrium.

- Since $s_j^*(v_j) = a_j + c_j v_j \in [a_j, a_j + c_j]$, we can restrict our attention to $b_i \in [a_j, a_j + c_j]$ (i.e., $b_i < a_j$ is pointless, while $b_i > a_j + c_j$ is not rational).
- Under the above restriction, we know that

$$0 \le \frac{b_i - a_j}{c_j} \le 1.$$

- Player i's best response b_i solves

$$\max_{a_j \le b_i \le a_j + c_j} (v_i - b_i)\frac{b_i - a_j}{c_j}.$$

FOC w.r.t. b_i.

- The best response of player i is

$$s_i(v_i) = \begin{cases} a_j, & \text{if } v_i \le a_j; \\ \frac{1}{2}(v_i + a_j), & \text{if } a_j < v_i \le a_j + 2c_j; \\ a_j + c_j, & \text{if } v_i > a_j + 2c_j. \end{cases}$$

不能花的比vi更多.

$\frac{1}{2}(v_i + a_j) < v_i \Rightarrow a_j < v_i$

$\frac{1}{2}(v_i + a_j) \le a_j + c_j \Rightarrow v_i \le a_j + 2c_j$

↓

从 $b_i \le a_j + c_j$ 得出.

$\quad \| $

$\frac{1}{2}(v_i + a_j)$

- We want the equilibrium bid to be a linear function on $[0, 1]$.
- There are three cases:

$$[0, 1] \subseteq \begin{cases} (-\infty, a_j] \\ [a_j, a_j + 2c_j] \\ [a_j + 2c_j, \infty) \end{cases}$$

- Case 1 violates the assumption $a_j < 1$.
- Case 3 violates the assumptions $a_j \ge 0$ and $c_j > 0$, which imply $a_j + 2c_j > 0$.
- Therefore, we have $[0, 1] \subseteq [a_j, a_j + 2c_j]$, and the best response is

$$s_i(v_i) = \frac{1}{2}(v_i + a_j).$$

- In a Bayesian Nash equilibrium,

$$s_i^*(v_i) = a_i + c_i v_i = \frac{1}{2}(v_i + a_j)$$

for all $v_i \in [0, 1]$.

- Then we have

$$a_i = \frac{1}{2}a_j, \text{ and } c_i = \frac{1}{2}$$

for $i, j = 1, 2$ and $i \neq j$.

- Therefore

$$a_1 = a_2 = 0, \text{ and } c_1 = c_2 = \frac{1}{2}.$$

- The unique linear Bayesian Nash equilibrium is

$$s_1^*(v_1) = \frac{1}{2}v_1, \text{ and } s_2^*(v_2) = \frac{1}{2}v_2.$$

(不考) (见 Slides)

- A Double Auction

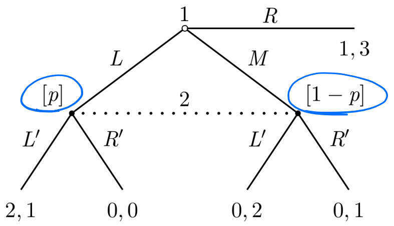
Lec 12. Perfect Bayesian Equilibrium.

· Requirements

以多大的概率在不同的 node 上.

1) At each information set, the player with the move must have a **belief** about which node in the information set has been reached by the play of the game. For a nonsingleton information set, a belief is a probability distribution over the nodes in the information set; for a singleton information set, a belief puts probability one on the single decision node.

- In Example 1, Requirement 1 implies that if player 2's nonsingleton information set is reached, player 2 must form a belief on which of the decision node has been reached, i.e., player 2 believes that player 1 has chosen L with probability p, and M with probability $1 - p$, where $p \in [0, 1]$.



序贯理性

2) Given their beliefs, the players' strategies must be **sequentially rational**. That is, at each information set, the action taken by the player with the move (and the player's subsequent strategy) must be optimal, given the player's belief at that information set and the other players' subsequent strategies (where a "subsequent strategy" is a complete plan of action covering every contingency that might arise after the given information set has been reached).

- Given this belief, player 2's expected payoffs are
  - playing L': $p \cdot 1 + (1 - p) \cdot 2 = 2 - p$
  - playing R': $p \cdot 0 + (1 - p) \cdot 1 = 1 - p$
- Since R' is never optimal for any belief, (R, R') cannot satisfy Requirement 2.
- Requirements 1 and 2 together can already eliminate the equilibrium (R, R') which relies on a non-credible threat.
- Requirements 1 and 2 allow for arbitrary beliefs, including unreasonable ones. Further requirements on players' beliefs need to be introduced.

Def. For a given equilibrium in a given extensive-form game, an information set is **on the equilibrium path** if it will be reached with positive probability if the game is played according to the equilibrium strategies, and is **off the equilibrium path** if it is definitely not to be reached if the game is played according to the equilibrium strategies.

- In Example 1, player 1's singleton information set is always on the equilibrium path.
- Consider player 2's nonsingleton information set.
- For the equilibrium (L, L'), the nonsingleton information set is on the equilibrium path.
- For the equilibrium (R, R'), the nonsingleton information set is off the equilibrium path.

3) At information sets on the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies.

(条件概率)

- In Example 1, for the equilibrium (L, L'), Requirement 3 implies that player 2's belief must be $p = 1$.

结构类似example 1. 概率假定如下.

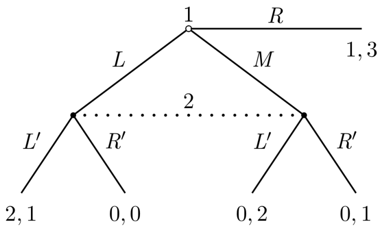
- Consider a hypothetical situation: the game has a mixed-strategy equilibrium in which player 1 plays L with probability q_1, M with probability q_2, and R with probability $1 - q_1 - q_2$. Requirement 3 would force player 2's belief to be

原因: L和M在同一信息集中.

$$p = \text{Prob}(L \text{ is played } | L \text{ or } M \text{ is played}) = \frac{q_1}{q_1 + q_2}.$$

· lead in example

- Example 1:



- What are the pure-strategy Nash equilibria and subgame-perfect Nash equilibria in this game?
- The normal-form representation of the game is

		Player 2	
		L'	R'
Player 1	L	2, 1	0, 0
	M	0, 2	0, 1
	R	1, 3	1, 3

- Two pure-strategy Nash equilibria:

$$(L, L') \text{ and } (R, R')$$

- Since the above game has no subgames, both (L, L') and (R, R') are subgame-perfect Nash equilibria.
- However, (R, R') is based on a non-credible threat from player 2.
- On the one hand, if player 1 believes player 2's threat of playing R', then player 1 should choose R to end the game with payoff 1, which is larger than 0 by choosing L or M.
- On the other hand, if player 1 doesn't believe the threat and plays L or M, then when player 2 gets the move, he will indeed choose L', since L' is strictly better than R' for player 2.
- Thus, the threat of playing R' by player 2 is not credible.
- In Example 1, the equilibrium (R, R') is not reasonable as it depends on a non-credible threat.
- We need to strengthen the equilibrium concept to rule out some subgame-perfect Nash equilibria like (R, R').
- A stronger equilibrium concept ⇒ **perfect Bayesian equilibrium**

- Here the "equilibrium" can mean Nash equilibrium, subgame-perfect Nash equilibrium, Bayesian Nash equilibrium or perfect Bayesian equilibrium.

→ 具体操作: 对于每个子博弈精炼NE. 看其中是否含有可到达的信息集.
再用 Requirment 3/4.

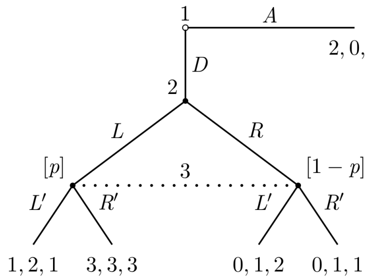
4) At information sets off the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies where possible.

- In Example 1, for the equilibrium (R, R'), Requirement 4 does not put any restrictions on player 2's belief p.

Def. A **perfect Bayesian equilibrium** consists of strategies and beliefs satisfying Requirements 1 through 4.

e.g.

- Example 2:



- What are the (pure-strategy) Nash equilibria and subgame-perfect Nash equilibria of this game? Are they also perfect Bayesian equilibria?
- The normal-form representation of the game:

	L	R
A	2, 0, 0	2, 0, 0
D	1, 2, 1	0, 1, 2

Player 3 chooses L'

	L	R
A	2, 0, 0	2, 0, 0
D	3, 3, 3	0, 1, 1

Player 3 chooses R'

- Player 1 chooses the row, player 2 chooses the column and player 3 chooses the matrix.
- Four pure-strategy Nash equilibria:

$$(A, L, L'), (A, R, L'), (A, R, R'), \text{ and } (D, L, R')$$

- The game has a unique subgame (beginning at player 2's singleton information set), and the unique Nash equilibrium of this subgame is (L, R').
- Hence, the unique subgame-perfect Nash equilibrium of the game is (D, L, R').
- The other three Nash equilibria are not subgame-perfect.
- Check whether each equilibrium is a perfect Bayesian equilibrium. → 依次验证 requirment 1-4.

Hint: 用 requirment 2 可解出 p 的临界值 $\frac{1}{3}$.

用 requirment 4 可解出 $p=1$.

⇒ (A, L, L′) 舍 ($p \leq \frac{1}{3}$ 企 $p=1$).

- Consider the subgame-perfect Nash equilibrium (D, L, R').
- These strategies and the belief $p = 1$ for player 3 satisfy Requirements 1-3.
- They also satisfy Requirement 4, since there is no information set off the equilibrium path
- Then the strategies (D, L, R') and the belief $p = 1$ indeed constitute a perfect Bayesian equilibrium.
- The other three Nash equilibria do not satisfy all Requirements 1-4.
- For example, consider the Nash equilibrium (A, L, L').
- Requirement 4 implies that for player 3's nonsingleton information set off the equilibrium path, player 3's belief must be $p = 1$.
- Requirement 2 then implies that for $p = 1$, player 3 must choose R' rather than L'.
- Therefore, the strategies (A, L, L') and the belief $p = 1$ do not satisfy Requirements 1 to 4, and they are not a perfect Bayesian equilibrium.

☆Note: 例1不可用而例2可用的原因: 例2中. player 2 "搅"了一下. i.e. player 2 有正概率选 L/R. ⇒ 用 Bayes' 法则不会出 $\frac{0}{0}$.

例1中. $p=\frac{L}{L+M}=\frac{0}{0}$.

例2中. $p=\frac{L}{L+R}=1$.

在例2中. 检验 (A, L, L′).

若1选D (实际上不可能发生).

则可用 Bayes' 法则. $p=\frac{1}{1}=1$.

↓

理解: $p=\frac{\varepsilon_1(1-\varepsilon_2)}{\varepsilon_1(1-\varepsilon_2)+\varepsilon_1\varepsilon_2}$.

ε_1: P(1选D)

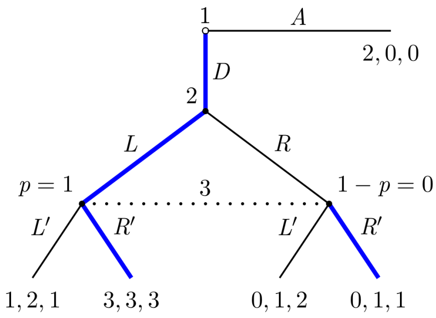
ε_2: P(2选R)

$\varepsilon_1, \varepsilon_2 \to 0$.

☆Note. 解出来的 p 要与均衡策略一致.

belief ↑ 实际 ↑

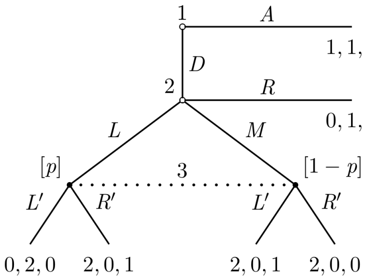
e.g. 不能解出 $p \leq \frac{1}{3}$ 但实际上 Player 2 一定实行 L.



Perfect Bayesian equilibrium in Example 2: $((D, L, R'); p = 1)$

e.g.

- Example 3:



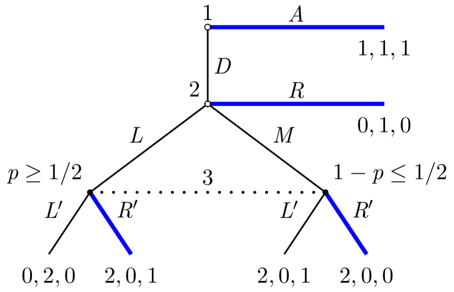
- Three pure-strategy Nash equilibria:

$$(A, L, L'), (A, R, L'), \text{ and } (A, R, R')$$

①
- Consider the strategies (A, L, L') and the belief $p \leq 1/2$, which satisfy Requirements 1 to 3.
- Requirement 4 implies that for player 3's information set off the equilibrium path, the belief must be $p = 1$, which contradicts $p \leq 1/2$.
- Therefore, there exists no belief together with the strategies (A, L, L') that constitutes a perfect Bayesian equilibrium.

②
- Consider strategies (A, R, L') and the belief $p \leq 1/2$.
- They satisfy Requirement 4, which puts no restrictions on player 3's belief at the information set off the equilibrium path.
- They also satisfy Requirements 1 and 3.
- However, at player 2's singleton information set, player 2 should choose L rather than R given player 3's equilibrium strategy, which implies that Requirement 2 is violated.
- Thus, strategies (A, R, L') and the belief $p \leq 1/2$ do not constitute a perfect Bayesian equilibrium.

③
- Consider the strategies (A, R, R') and the belief $p \geq 1/2$.
- They satisfy all Requirements 1-4, and thus constitute a perfect Bayesian equilibrium.



Perfect Bayesian equilibrium in Example 3: $((A, R, R'); p \geq 1/2)$

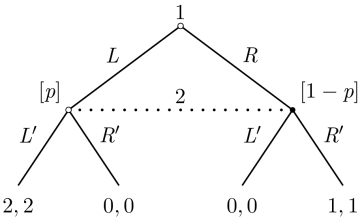
☆
- The procedure to determine whether a given equilibrium is a perfect Bayesian equilibrium:
  1. (1) Determine a belief for each information set by Bayes' rule;
  2. (2) Check whether the equilibrium is optimal given each belief determined in (1) and the subsequent strategies.
- A perfect Bayesian equilibrium consists not only strategies but also beliefs of players, and it requires each player's strategy to be optimal given his or her reasonable beliefs.

· Relationship between Different Equilibrium Concepts

- Perfect Bayesian equilibrium is a stronger equilibrium concept that 精炼 refines different types of equilibria.
- On the one hand, it refines Bayesian Nash equilibrium (in the same way as subgame-perfect Nash equilibrium refines Nash equilibrium).
- On the other hand, it strengthens subgame-perfect Nash equilibrium by explicitly analyzing beliefs.
- In addition, while a Nash equilibrium requires that no player chooses a strictly dominated strategy, a perfect Bayesian equilibrium requires no player's strategy to be strictly dominated beginning at any information set.
- Perfect Bayesian equilibrium corresponds to
  - Nash equilibrium (with appropriate beliefs) in static games of complete information;
  - Bayesian Nash equilibrium in static games of incomplete information;
  - subgame-perfect Nash equilibrium (with appropriate beliefs) in dynamic games of complete and perfect information (and also many dynamic games of complete but imperfect information).

e.g.

- Example 4:



- Three perfect Bayesian equilibria:

$$((L, L'); p = 1), ((R, R'); p = 0),$$
$$\text{and} \quad \left(\left(\frac{1}{3}L + \frac{2}{3}R, \frac{1}{3}L' + \frac{2}{3}R'\right); p = 1/3\right)$$

- The normal-form representation of the game is

		Player 2	
		L'	R'
Player 1	L	2, 2	0, 0
	M	0, 0	1, 1

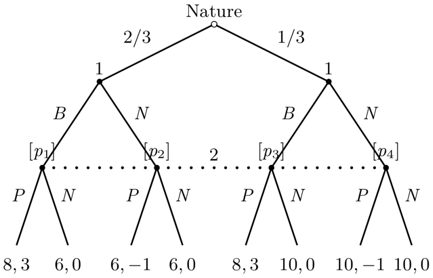
- Three Nash equilibria:

$$(L, L'), (R, R'), \text{ and } \left(\frac{1}{3}L + \frac{2}{3}R, \frac{1}{3}L' + \frac{2}{3}R'\right)$$

- Each Nash equilibrium (together with a correct belief) corresponds to a perfect Bayesian equilibrium in this static game of complete information.

e.g.

- Example 5:



- Two (pure-strategy) Bayesian Nash equilibria: (BN, P) and (NN, N)
- Two (pure-strategy) perfect Bayesian equilibria:

$$((BN, P); p_1 = 2/3, p_4 = 1/3), ((NN, N); p_2 = 2/3, p_4 = 1/3)$$

- Consider the first equilibrium, for example.
- For the strategy BN chosen by player 1, Requirement 3 implies that the belief is $p_1 = 2/3$ and $p_4 = 1/3$.
- Given this belief, it is optimal for player 2 to choose P.
- Given player 2's strategy P, it is optimal for player 1 type 1 to choose B, and type 2 to choose N.

Lec 13 Signaling Games

{ dynamic
incomplete information

Signaling → strategic models where informed agents take some observable actions before uninformed agents make their strategic decisions

- Signaling games are a relatively simple setting in which to study
  - how players update beliefs based on observed actions (signals);
  - how players try to strategically reveal or conceal private information by their choice of actions.

· Signaling Games

- A simple signaling game is a dynamic game of incomplete information involving two players: a Sender (S) (有私人信息) and a Receiver (R). (没有私人信息)
- The timing of the game is as follows:
  1. Nature draws a type t_i for the Sender from a set of feasible types $T = \{t_1, \dots, t_I\}$ according to a probability distribution $P(t_i)$, where $P(t_i) > 0$ for every i and $P(t_1) + \cdots + P(t_I) = 1$.
  2. The Sender observes t_i and then chooses a message m_j from a set of feasible messages $M = \{m_1, \dots, m_J\}$.
  3. The Receiver observes m_j (but not t_i) and then chooses an action a_k from a set of feasible actions $A = \{a_1, \dots, a_K\}$.
  4. Payoffs are given by $U_S(t_i, m_j, a_k)$ and $U_R(t_i, m_j, a_k)$.

- Consider the following signaling game:

$$T = \{t_1, t_2\}, A = \{a_1, a_2\}, P(t_1) = p, \text{ and } M = \{m_1, m_2\}.$$

- The Sender has four pure strategies:

$$(m_1, m_1), (m_1, m_2), (m_2, m_1), \text{ and } (m_2, m_2).$$

- The strategy (m', m'') means the Sender of type t_1 chooses a message m' and type t_2 chooses a message m''.
- Similarly, the Receiver has four pure strategies:

$$(a_1, a_1), (a_1, a_2), (a_2, a_1), \text{ and } (a_2, a_2).$$

- The strategy (a', a'') means the Receiver plays a' if the Sender chooses m_1 and plays a'' if the Sender chooses m_2.
- We call Sender's strategies (m_1, m_1), (m_2, m_2) to be pooling (混同策略) (because each type sends the same message), and (m_1, m_2), (m_2, m_1) to be separating (分离策略) (because each type sends a different message).

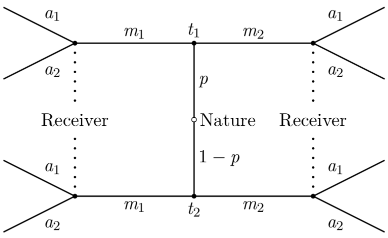


Figure 1: A signaling game

· Signaling Requirements

1) After observing any message m_j from M, the Receiver must have a belief about which types could have sent m_j. Denote this belief by the probability distribution $\mu(t_i|m_j)$, where $\mu(t_i|m_j) \geq 0$ for each $t_i \in T$, and $\sum_{t_i \in T} \mu(t_i|m_j) = 1$.

↱ 序贯理性.

2) Receiver: (仅依赖自己的 u 和 belief).

For each $m_j \in M$, the Receiver's action $a^*(m_j)$ must maximize the Receiver's expected utility, given the belief $\mu(t_i|m_j)$ about which types could have sent m_j. That is, $a^*(m_j)$ solves

$$\max_{a_k \in A} \sum_{t_i \in T} \mu(t_i|m_j) U_R(t_i, m_j, a_k).$$

Sender: (依赖 Receiver 的策略)

For each $t_i \in T$, the Sender's message $m^*(t_i)$ must maximize the Sender's utility, given the Receiver's strategy $a^*(m_j)$. That is, $m^*(t_i)$ solves

$$\max_{m_j \in M} U_S(t_i, m_j, a^*(m_j)).$$

- These two requirements imply that both the Receiver and the Sender act in an optimal way.
- Given the Sender's optimal strategy $m^*(t_i)$, i.e., m^* is a function from T into M, let $T_j = \{t_i \in T : m^*(t_i) = m_j\}$. T_j is the set of all types sending the message m_j.
- The information set corresponding to m_j is on the equilibrium path if $T_j \neq \emptyset$, and off the equilibrium path otherwise.

3) For each $m_j \in M$, if there exists $t_i \in T$ such that $m^*(t_i) = m_j$, i.e., $T_j \neq \emptyset$, then the Receiver's belief at the information set corresponding to m_j must follow from Bayes' rule and the Sender's strategy:

$$\mu(t_i|m_j) = \frac{P(t_i)}{\sum_{t \in T_j} P(t)}, \forall t_i \in T_j.$$

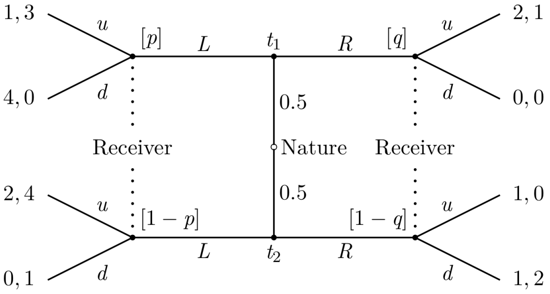
- Perfect Bayesian Equilibria

Def. A pure-strategy **perfect Bayesian equilibrium** in a signaling game is a pair of ①strategies $m^*(t_i)$ and $a^*(m_j)$ and ②a belief $\mu(t_i|m_j)$ satisfying Signaling Requirements (1), (2R), (2S), and (3).

- A strategy for the Sender is a function from the type space T into the message space M; a strategy for the Receiver is a function from the message space M into the action space A.
- For a perfect Bayesian equilibrium of a signaling game, if the Sender's strategy is pooling (or separating), then we call the equilibrium pooling (or separating), respectively.

e.g.

- Find all pure-strategy perfect Bayesian equilibria in the following signaling game.



- The first (the second) number is the payoff to the Sender (the Receiver).
- In this game,

$$T = \{t_1, t_2\}, P(t_1) = 0.5, M = \{L, R\}, A = \{u, d\}.$$

- The Sender's strategies are: (L, L), (L, R), (R, L) and (R, R), where (m', m'') means that type t_1 chooses m' and type t_2 chooses m''.
- The Receiver's strategies are: (u, u), (u, d), (d, u), and (d, d), where (a', a'') means that the Receiver plays a' following L and a'' following R.
- We analyze the possibility of the four Sender's strategies to constitute perfect Bayesian equilibria.
- **Case 1: Pooling on L**
- Suppose the Sender adopts the strategy (L, L).
- By Signaling Requirement 3, we have $p = 1 - p = 0.5$. Given this belief (or any belief) of the Receiver, the Receiver's best response to message L is u, i.e., $a^*(L) = u$.

↗原因：均衡路径之外.

- For the message R, the Receiver's belief q cannot be determined by Sender's strategy, and thus we can choose any belief q. Furthermore, both $a^*(R) = u$ and $a^*(R) = d$ are possible for some q. Indeed $a^*(R) = u$ iff $q \geq 2/3$; and $a^*(R) = d$ iff $q \leq 2/3$.

基于Receiver的choice给出的限制.

- We only need to see if sending L is better than sending R for both types t_1 and t_2.
- If $a^*(R) = u$, i.e., (u, u) is the Receiver's strategy, then for type t_1, the Sender's payoff is 1 if L is sent and 2 if R is sent. Hence, sending L is not optimal.
- If $a^*(R) = d$, i.e., (u, d) is the Receiver's strategy, then for type t_1, the Sender's payoff is 1 if L is sent and 0 if R is sent, choosing L is optimal; for type t_2, choosing L is also optimal given $2 > 1$.

给定Sender的type条件下看选择是否理性.

- Thus, (L, L) is the Sender's best response to the Receiver's strategy (u, d).
- Moreover, (u, d) is also the Receiver's best response to the Sender's strategy (L, L) if $q \leq 2/3$.
- Therefore, $[(L, L), (u, d); p = 0.5, q \leq 2/3]$ is a pooling equilibrium.

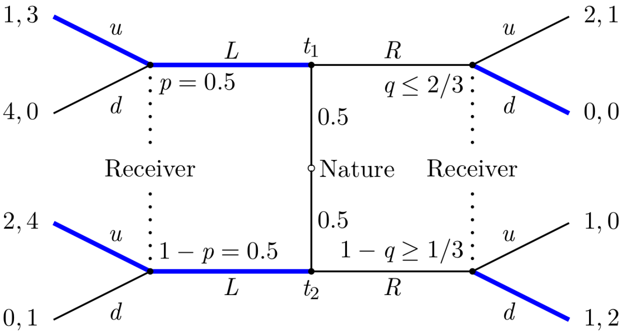


Figure 2: Pooling equilibrium: $[(L, L), (u, d); p = 0.5, q \leq 2/3]$

- **Case 2: Pooling on R**
- Suppose the Sender adopts the strategy (R, R).
- Then Signaling Requirement 3 implies that $q = 1 - q = 0.5$. Given this belief, the Receiver's best response to R is d, i.e., $a^*(R) = d$, since $0.5 < 1$.
- For the message L, we can choose any belief p. But we know for any p, the Receiver's best response to L is u, i.e., $a^*(L) = u$.
- Given the Receiver's strategy (u, d), for type t_1, the Sender's payoff is 0 if R is sent and 1 if L is sent, and thus R is not optimal.
- Therefore, there is no equilibrium in which the Sender plays (R, R).
- **Case 3: Separation with t_1 playing L**
- Suppose the Sender adopts the separating strategy (L, R).
- Then, Signaling Requirement 3 implies $p = 1$ and $q = 0$. For these beliefs, we must have $a^*(L) = u$, and $a^*(R) = d$.
- Given the Receiver's strategy (u, d), for type t_2, the Sender's payoff is 2 if L is sent and 1 if R is sent. Hence R is not optimal.
- Therefore, there is no equilibrium in which the Sender plays (L, R).
- **Case 4: Separation with t_1 playing R**
- Suppose the Sender adopts the separating strategy (R, L).
- Then, Signaling Requirement 3 implies $p = 0$ and $q = 1$. For these beliefs, we have $a^*(L) = u$ and $a^*(R) = u$.
- Given the Receiver's strategy (u, u), for type t_1, the Sender's payoff is 1 if L is sent and 2 if R is sent. Hence R is optimal.
- For the Sender type t_2, the payoff is 2 if L is sent and 1 if R is sent. Hence L is also optimal.
- Therefore, $[(R, L), (u, u); p = 0, q = 1]$ is a separating perfect Bayesian equilibrium.

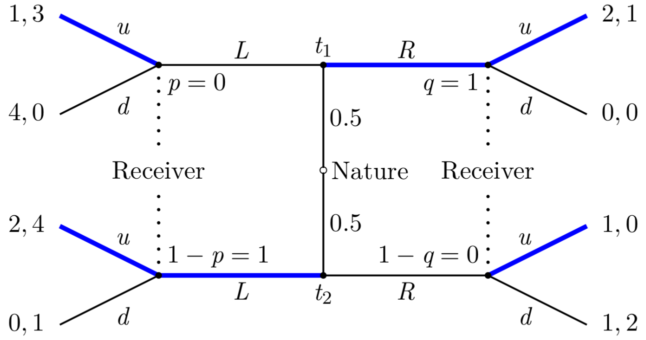


Figure 3: Separating equilibrium: $[(R, L), (u, u); p = 0, q = 1]$

- How to find (pure-strategy) perfect Bayesian equilibria in signaling games:
  (1) Start with a strategy of the Sender (pooling or separating);
  (2) If possible, calculate the beliefs of the Receiver using Bayes' rules. Otherwise, choose arbitrary beliefs; (Requirement 3)
  (3) Given the beliefs, find out the best response of the Receiver; (Requirement 2)
  (4) Check whether the Sender's strategy is a best response to the Receiver's strategy.

· 法2

- Consider an alternative way to find perfect Bayesian equilibria.
- We first find Bayeisan Nash equilibria, and then check which equilibria are perfect Bayesian equilibria.
- Consider the following matrix to represent the game:

		Receiver			
		(u, u)	(u, d)	(d, u)	(d, d)
Sender	(L, L)	$1, \underline{2}, \underline{3.5}$	$\underline{1}, \underline{2}, \underline{3.5}$	$\underline{4}, 0, 0.5$	$\underline{4}, 0, 0.5$
	(L, R)	$1, 1, 1.5$	$\underline{1}, 1, \underline{2.5}$	$\underline{4}, \underline{1}, 0$	$\underline{4}, \underline{1}, 1$
	(R, L)	$\underline{2}, \underline{2}, \underline{2.5}$	$0, \underline{2}, 2$	$2, 0, 1$	$0, 0, 0.5$
	(R, R)	$\underline{2}, 1, 0.5$	$0, 1, \underline{1}$	$2, \underline{1}, 0.5$	$0, \underline{1}, \underline{1}$

- Two (pure-strategy) Bayesian Nash equilibria: $((L, L), (u, d))$ and $((R, L), (u, u))$
- To check whether they are perfect Bayesian equilibria, we only need to find beliefs, satisfying all four Signaling Requirements.
- For (L, L), Bayes' rule requires $p = 0.5$ and there is no requirement for q. Given the belief, $a^*(L) = u$, and $a^*(R) = d$ iff $q \leq 2/3$. Thus (u, d) is a best response to (L, L) iff $p = 0.5$ and $q \leq 2/3$.
- For (R, L), Bayes' rule requires $p = 0$ and $q = 1$. Given this belief, $a^*(L) = u$ and $a^*(R) = u$. Thus (u, u) is a best response to (R, L).
- Therefore, $[(L, L), (u, d); p = 0.5, q \leq 2/3]$ and $[(R, L), (u, u); p = 0, q = 1]$ are two perfect Bayesian equilibria.

混合/杂合均衡

· Hybrid Equilibria

☆区别于混同策略 → 不同t_i下策略相同
↳pooling

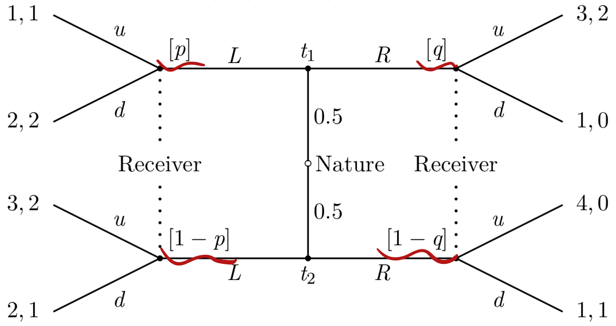
↳ 不同t_i下依概率决定策略.

→ pooling: 一个状态对应一种确定的选择.

hybrid: 一个状态对应一个策略组合.

↳ P_i的概率 play s_i.

- Consider the following signaling game.



- Besides pure-strategy perfect Bayesian equilibria (either separating or pooling), there may exist hybrid equilibria in which some type of the Sender randomizes.

e.g.

- Consider a hybrid equilibrium in which type t_1 Sender randomizes between L and R, while type t_2 Sender chooses R.
- Let the hybrid strategy of the Sender being $(1 - r_1)L + r_1 R$ for type t_1, where $0 < r_1 < 1$, and R for type t_2.
- Given the Sender's strategy, the Receiver's beliefs are $p = 1$ and $q = \frac{r_1}{1+r_1}$ by Bayes' rules. → 收到L则一定是t_1 因为t_2不会发L.
- ① Then the Receiver's best response to message L is d.
- ② For message R, given the belief q, the Receiver's best response is

$$\beta_R(q) = \begin{cases} u, & \text{if } q > \frac{1}{3}; \\ \overset{\text{任意.}}{r_2} u + (1 - r_2)d, & \text{if } q = \frac{1}{3}; \rightarrow u与d无差异. \\ d, & \text{if } q < \frac{1}{3}. \end{cases}$$

- If the Receiver chooses u, then type t_1 Sender would choose R rather than $(1 - r_1)L + r_1 R$.
- If the Receiver chooses d, then type t_1 Sender would choose L rather than $(1 - r_1)L + r_1 R$.
- The remaining possibility is for the Receiver to choose $r_2 u + (1 - r_2)d$ for $0 < r_2 < 1$, which requires that $q = \frac{1}{3}$.
- This further implies that $r_1 = \frac{1}{2}$. $q = \frac{r_1}{1+r_1}$
- Given that the Receiver chooses $r_2 u + (1 - r_2)d$ when R is sent, where $0 < r_2 < 1$, type t_1 Sender will choose $\frac{1}{2}L + \frac{1}{2}R$ if

$$2 = 3r_2 + (1 - r_2).$$ → t_1状态下 左=右 (期望)

  Hence, we get $r_2 = \frac{1}{2}$.
- Given the Receiver's choice, type t_2 Sender gets an expected payoff of $\frac{5}{2}$ when choosing R, which is strictly higher than 2.
- Therefore, it is indeed optimal for type t_2 Sender to choose R.
- In sum, the following is a hybrid equilibrium: $[(\frac{1}{2}L + \frac{1}{2}R, R), (d, \frac{1}{2}u + \frac{1}{2}d); (p = 1, q = \frac{1}{3})]$.

(了解即可)

· Cheap - Talk Games

- Cheap-talk games are analogous to signaling games, but the Sender's messages are just talk, i.e., costless, non-binding, nonverifiable claims.
- Cheap talk cannot be informative in some cases (for example, Spence's job-market signaling model).
- There are situations where cheap talk can convey some information (although may not be fully precise), for example, Stein (1989), Matthews (1989), Austen-Smith (1990).
- In general, cheap talk can be informative under certain conditions.

- The timing of the simplest cheap-talk game is identical to the timing of the simplest signaling game (only payoff functions differ):
  1. Nature draws a type t_i for the Sender from a set of feasible types $T = \{t_1, \ldots, t_I\}$ according to a probability distribution $P(t_i)$, where $P(t_i) > 0$ for every i and $P(t_1) + \cdots + P(t_I) = 1$.
  2. The Sender observes t_i and then chooses a message m_j from a set of feasible messages $M = \{m_1, \ldots, m_J\}$.
  3. The Receiver observes m_j (but not t_i) and then chooses an action a_k from a set of feasible actions $A = \{a_1, \ldots, a_K\}$.
  4. Payoffs are given by $U_S(t_i, a_k)$ and $U_R(t_i, a_k)$.

- The key feature of the cheap-talk game is that the message has no direct effect on the payoffs of the Sender and the Receiver.
- The message can only be informative by changing the Receiver's belief about the Sender's type.
- Since anything can be said (i.e., M can be a very large set), it is typically assumed that $M = T$.
- The definition of perfect Bayesian equilibrium in a cheap-talk game is identical to that in a signaling game.
- One key difference between these two games is that there always exists a pooling equilibrium in a cheap-talk game.
- The following is a pooling equilibrium:

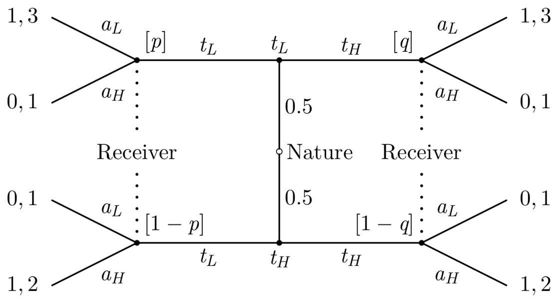
  $$m^*(t_i) = t^*, a^*(m_j) = a^*, \mu(t_i|m_j) = P(t_i)$$

  for all $t_i \in T$ and $m_j \in M$, where t^* is any message, and a^* solves

  $$\max_{a_k \in A} \sum_{t_i \in T} P(t_i)\, U_R(t_i, a_k).$$

- In this pooling equilibrium, the Sender of all types sends the same message t^*, while the Receiver keeps the prior belief of all messages and takes an action optimally according to the belief.
- An interesting question is whether there exists any non-pooling equilibrium in which communication can be effective.

- Find all pure-strategy perfect Bayesian equilibria of the following signaling game.



- Note that the above signaling game is indeed a cheap-talk game, since neither the Sender's payoff nor the Receiver's payoff depends on the messages.
- Clearly, there are two pooling equilibria:

  $$[(t_L, t_L), (a_L, a_L); p = 0.5, q \geq 1/3],$$

  and

  $$[(t_H, t_H), (a_L, a_L); p \geq 1/3, q = 0.5].$$

- There also exist two separating equilibria:

  $$[(t_L, t_H), (a_L, a_H); p = 1, q = 0],$$

  and

  $$[(t_H, t_L), (a_H, a_L); p = 0, q = 1].$$

- Consider a two-type, two-action example:

  $$T = \{t_L, t_H\}, P(t_L) = p, A = \{a_L, a_H\}, M = T.$$

- We use the following matrix to represent the payoffs: the first (second) number is the payoff to the Sender (Receiver).

	t_L	t_H
a_L	$x, 1$	$y, 0$
a_H	$z, 0$	$w, 1$

- Note that the above matrix differs from the normal-form representation of the game.
- Consider the following separating equilibrium:
  - the Sender's strategy: $[m^*(t_L) = t_L, m^*(t_H) = t_H]$;
  - the Receiver's beliefs: $\mu(t_L|t_L) = 1$ and $\mu(t_L|t_H) = 0$;
  - the Receiver's strategy: $[a^*(t_L) = a_L, a^*(t_H) = a_H]$.
- In the above equilibrium, each type of the Sender tells the truth.
- It can be shown that the separating equilibrium exists iff $x \geq z$ and $y \leq w$.
- In other words, the Sender's and the Receiver's interests perfectly align.
- In general, Crawford and Sobel (1982) have shown that more communication can occur through cheap talk when players' preferences are more closely aligned.